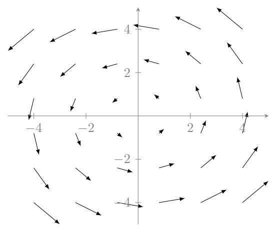
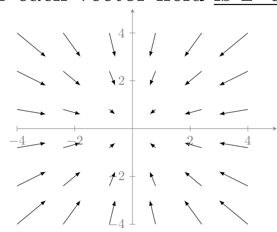
Math 2130 **Test #3** **April 14th, 2021**

Name: Answer Key **Be sure to show your work!**

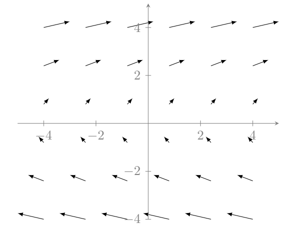
1. (12 points) The following are plots of several vector fields. Please note that all of the vectors have been scaled down so they do not overlap each other. Write A, B, and C next to the appropriate vector field's formula. Put an X next to the formula whose vector field is **not** shown. Also, for each vector field is **F** conservative? Circle "Yes" or "No".



A



B



C

C $\mathbf{F}(x,y) = \langle y, 1\rangle$ Yes / **No**

X $\mathbf{F}(x,y) = \langle x, y\rangle$ **Yes** / No

B $\mathbf{F}(x,y) = \langle -x, -y\rangle$ **Yes** / No

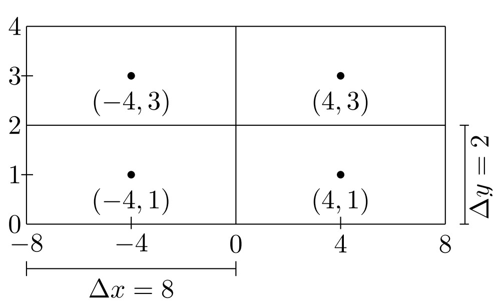
A $\mathbf{F}(x,y) = \langle -y, x\rangle$ Yes / **No**

To match up formulas, notice that in the first plot the arrow around $(1,1)$ points up and left. Plugging $(1,1)$ into the formulas yields: $\langle 1,1\rangle$, $\langle 1,1\rangle$, $\langle -1,-1\rangle$, and $\langle -1,1\rangle$ respectively. Only the last formula matches. The middle plot's vector at $(1,1)$ points down and left. Only the third formula matches. Finally, the last plot's arrows are all tilted upward (at least a little). This would be true of $\mathbf{F} = \langle y,1\rangle$ but not of $\mathbf{F} = \langle x,y\rangle$ (since this second formula would yield arrows pointing downward when $y<0$). Thus the last plot goes with the first formula. The second formula's vector field was not plotted (although it looks like plot B with the arrows reversed).

Next, recall that for a vector field, $\mathbf{F} = \langle M, N\rangle$ defined on all of $\mathbb{R}^2$, $\mathbf{F}$ is conservative if and only if $M_y = N_x$. If we have $\mathbf{F} = \langle y,1\rangle$, then $M=y$ and $N=1$ so $M_y = 1 \neq 0 = N_x$. Thus $\mathbf{F} = \langle y,1\rangle$ is not conservative. For the remaining formulas out calculations are: $M_y = 0 = N_x$, $M_y = 0 = N_x$, and $M_y = -1 \neq 1 = N_x$ respectively. Thus the middle vector fields are conservative but the last one is not.

2. (10 points) Use a double Riemann sum to approximate $\iint_R y\cos(x^2+1)\,dA$ where $R = [-8,8]\times[0,4]$.

Use midpoint rule and a 2×2 grid of rectangles (2 across and 2 up) to partition R. (Don't worry about simplifying.)



Our rectangle R is $8-(-8)=16$ units across so dividing up into two pieces gives us $\Delta x = 16/2 = 8$. Likewise, R is 4 units tall so $\Delta y = 4/2 = 2$. It is easy then to specify partition points and mark down midpoints. Therefore,

$$\iint_R y\cos(x^2+1)\,dA \approx$$

$$\boxed{8\cdot 2\cdot\Big(1\cos((-4)^2+1) + 1\cos(4^2+1) + 3\cos((-4)^2+1) + 3\cos(4^2+1)\Big)}$$

3. (7 points) Let $\mathbf{F}(x,y,z) = \langle 10, xyz, e^x\rangle$ and C be the curve parameterized by $\mathbf{r}(t) = \langle t^2, t^4, t^6\rangle$ where $-7 \leq t \leq 9$.

Set up but **do not** evaluate the line integral: $\int_C \mathbf{F}\bullet d\mathbf{r}$.

Notice that $\mathbf{r}'(t) = \langle 2t, 4t^3, 6t^5\rangle$. Our parameterization $\mathbf{r}(t)$ tells us that $x=t^2$, $y=t^4$, and $z=t^6$. Therefore,

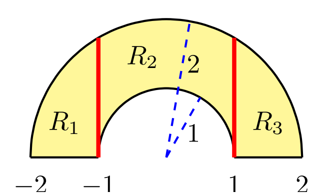
$$\int_C \mathbf{F}\bullet d\mathbf{r} = \int_{-7}^{9} \mathbf{F}(\mathbf{r}(t))\bullet\mathbf{r}'(t)\,dt = \int_{-7}^{9}\langle 10, t^2\cdot t^4\cdot t^6, e^{t^2}\rangle\bullet\langle 2t, 4t^3, 6t^5\rangle\,dt = \boxed{\int_{-7}^{9}\left(20t + 4t^{15} + 6t^5e^{t^2}\right)\,dt}$$

4. (13 points) Let R be the annular region between $x^2+y^2=1$ and $x^2+y^2=4$ where $y\geq 0$ (pictured below).

[*Warning:* One of the integrals below will have to be split into *several* pieces.]

(a) Set up the integral $\iint_R y\,dA$ in polar coordinates.

(b) Set up the integral $\iint_R y\,dA$ rectangular coordinates.



For polar coordinates, imagine a ray emanating from the origin. We enter the region when $r^2 = x^2 + y^2 = 1$ and exit when $r^2 = x^2 + y^2 = 4$. Thus $1 \le r \le 2$. Also, since $y \ge 0$, we need to restrict our angle: $0 \le \theta \le \pi$ (from the positive x-axis to the negative x-axis). Keeping in mind $y = r\sin(\theta)$ and the Jacobian $J = r$, we get that

$$\text{Answer (a):} \quad \iint_R y\,dA = \int_0^{\pi}\int_1^2 r\sin(\theta)\cdot r\,dr\,d\theta$$

For rectangular coordinates, while R is a y-simple region, it is not an x-simple region (because of the notch cut out of the bottom). Now, even though the region is y-simple, to set up the integral in rectangular coordinates we need to break up R into 3 pieces. Why? The bottom of the region changes formulas a few times. First, we have the x-axis: $y = 0$, then the upper half of the unit circle: $y = \sqrt{1-x^2}$, and finally the x-axis: $y = 0$ again. The top of R is easier. It is just the upper half of $x^2 + y^2 = 4$ (i.e., $y = \sqrt{4-x^2}$). Notice that the first piece corresponds to $-2 \le x \le -1$, the second piece to $-1 \le x \le 1$, and the third piece to $1 \le x \le 2$. Why? Just consider the radii of the circles.

$$\text{Answer (b):} \quad \iint_R y\,dA = \int_{-2}^{-1}\int_0^{\sqrt{4-x^2}} y\,dy\,dx + \int_{-1}^{1}\int_{\sqrt{1-x^2}}^{\sqrt{4-x^2}} y\,dy\,dx + \int_1^2\int_0^{\sqrt{4-x^2}} y\,dy\,dx$$

Alternatively, we could integrate over the upper half disk: $x^2 + y^2 \le 4$, $y \ge 0$ and remove the upper half unit disk: $x^2 + y^2 \le 1$, $y \ge 0$. Therefore,

$$\text{Alternate answer (b):} \quad \iint_R y\,dA = \int_{-2}^{2}\int_0^{\sqrt{4-x^2}} y\,dy\,dx - \int_{-1}^{1}\int_0^{\sqrt{1-x^2}} y\,dy\,dx$$

(c) Find the centroid of R.

Since R is symmetric across the y-axis (i.e., $x = 0$), we have that $\bar{x} = 0$. Also, $m = \iint_R 1\,dA$ is just the area of our annular region. Thus $m = \frac{\pi 2^2 - \pi 1^2}{2} = \frac{3\pi}{2}$. Therefore, we only need to compute $M_x = M_{y=0} = \iint_R y\,dA$. Obviously, part (a)'s set up is the best way to go:

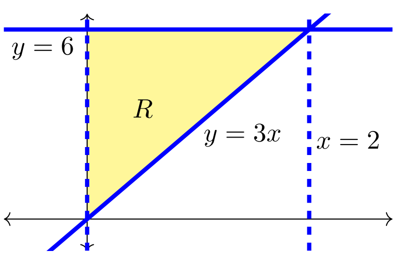
$$M_x = \iint_R y\,dA = \int_0^{\pi}\int_1^2 r^2\sin(\theta)\,dr\,d\theta = \int_0^{\pi}\sin(\theta)\,d\theta\int_1^2 r^2\,dr = 2\cdot\frac{r^3}{3}\Big|_1^2 = \frac{2}{3}(8-1) = \frac{14}{3}$$

Thus $\bar{y} = \dfrac{M_x}{m} = \dfrac{14/3}{3\pi/2} = \dfrac{28}{9\pi}$. Therefore, $\boxed{(\bar{x},\bar{y}) = \left(0, \dfrac{28}{9\pi}\right)}$.

5. (11 points) Consider $\displaystyle\int_0^2\int_{3x}^6 \sin(y^2)\,dy\,dx$.

(a) Sketch the corresponding region of integration.

(b) Compute the iterated integral. *Hint:* You cannot integrate $\int \sin(y^2)\,dy$ in terms of elementary functions.



Our bounds of integration are: $3x \le y \le 6$ and $0 \le x \le 2$. So we integrate from the line $y = 3x$ up to the horizontal line $y = 6$. This is done for x's ranging from $x = 0$ over to $x = 2$. Notice that $(x, y) = (2, 6)$ is where our lines intersect. Now we cannot integrate with respect to y first since $\sin(y^2)$ has no elementary anti-derivative. Thus we must *reverse the order of integration*. Notice that as an x-simple region, we go from the y-axis (i.e., $x = 0$) over to the line $y = 3x$ (i.e., $x = y/3$). Then we range from $y = 0$ up to $y = 6$. Thus our bounds are $0 \le x \le y/3$ and $0 \le y \le 6$. Our integral is:

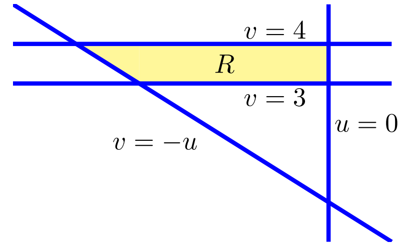
$$= \iint_R \sin(y^2)\,dA = \int_0^6\int_0^{y/3}\sin(y^2)\,dx\,dy = \int_0^6 x\sin(y^2)\Big|_0^{y/3} dy = \int_0^6 \frac{y}{3}\sin(y^2) - 0\,dy = \frac{-\cos(y^2)}{6}\Bigg|_0^6$$

where we used a simple u-substitution to integrate $y\sin(y^2)$ (i.e., $u = y^2$ and $du/2 = y\,dy$). Therefore, our integral evaluates to $\dfrac{-\cos(6^2)}{6} - \dfrac{-\cos(0^2)}{6} = \boxed{\dfrac{1-\cos(36)}{6}}$.

6. (11 points) Set up the integral $\displaystyle\iint_R e^{(x-y)^2}\,dA$ where R is the region bounded by the lines $y = x-3$, $y = x-4$, $y = -2x$, and $x = 0$. *Note:* Use the change of coordinates: $\boxed{u = 2x + y \text{ and } v = x - y}$. **DO NOT** evaluate this integral.

We need to transform our function (this is easy since $v = x - y$ so that $e^{(x-y)^2} = e^{v^2}$), transform our bounds, and compute our Jacobian. We have $\dfrac{\partial(u,v)}{\partial(x,y)} = \det\begin{bmatrix} u_x & u_y \\ v_x & v_y\end{bmatrix} = \det\begin{bmatrix} 2 & 1 \\ 1 & -1\end{bmatrix} = 2(-1) - 1(1) = -3$. But this is backwards since we should compute old variables in terms of new variables. Thus our Jacobian is $J = \dfrac{\partial(x,y)}{\partial(u,v)} = 1\Big/\dfrac{\partial(u,v)}{\partial(x,y)} = -1/3$.

Next, we need to transform our bounds: $x-y=3$, $x-y=4$, $2x+y=0$, and $x=0$. Since $u=2x+y$ and $v=x-y$, the first three are easy to transform: $v=3$, $v=4$, $u=0$. Our last bound says $x=0$ so that $u=2(0)+y$ and $v=0-y$. Thus $u=y=-v$. We can sketch these bounds and find that our integral is most easily set up as a u-simple region. Thus our region has bounds $-v \le u \le 0$ and $3 \le v \le 4$. Don't forget to take the absolute value of the Jacobian! We get that...

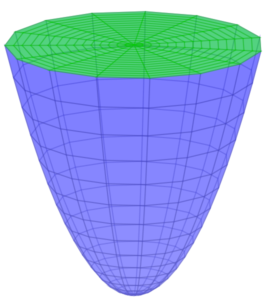


$$\iint_R e^{(x-y)^2}\,dA = \boxed{\int_3^4 \int_{-v}^0 e^{v^2}\cdot\frac{1}{3}\,du\,dv} = \int_3^4 \frac{1}{3}ue^{v^2}\Big|_{-v}^{0}\,dv = \int_3^4 0-\frac{1}{3}(-v)e^{v^2}\,dv$$

$$= \int_3^4 \frac{1}{3}ve^{v^2}\,dv = \frac{1}{6}e^{v^2}\Big|_3^4 = \frac{e^{16}-e^9}{6}$$

Note: I went ahead and integrated, but we only needed to set this up. The official answer is the boxed one.

7. (10 points) Let E be the region bounded by $z=x^2+y^2+1$ and $z=5$. Compute the integral $\iiint_E \frac{1}{\sqrt{x^2+y^2}}\,dV$.

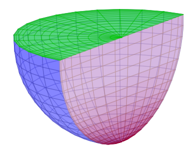


This region is bounded below by the circular paraboloid $z=x^2+y^2+1$ and above by the horizontal plan $z=5$. Considering our integrand (i.e., $1/\sqrt{x^2+y^2}$) and the bounds for this region, switching to cylindrical coordinates is an obvious choice. To that end we get bounds $z=r^2+1$ and $z=5$. Intersecting these bounds yields $r^2+1=5$ so that $r^2=4$ and thus $r=2$. Therefore, if we project out the z-direction, we are left with a disk of radius 2 centered at the origin. Therefore, E's bounds are $r^2+1\le z\le 5$, $0\le r\le 2$, and $0\le\theta\le 2\pi$. Our integrand becomes $1/r$ in cylindrical coordinates since $r=\sqrt{x^2+y^2}$. Finally, don't forget the Jacobian: r.

$$\iiint_E \frac{1}{\sqrt{x^2+y^2}}\,dV = \int_0^{2\pi}\int_0^2\int_{r^2+1}^5 \frac{1}{r}\cdot r\,dz\,dr\,d\theta = \int_0^{2\pi}d\theta\int_0^2\int_{r^2+1}^5 1\,dz\,dr$$

$$= 2\pi\int_0^2 z\Big|_{r^2+1}^5\,dr = 2\pi\int_0^2 5-(r^2+1)\,dr = 2\pi\int_0^2 4-r^2\,dr = 2\pi\left(4r-\frac{r^3}{3}\right)\Big|_0^2 = 2\pi\left(8-\frac{8}{3}\right) = \boxed{\frac{32}{3}\pi}$$

8. (12 points) Consider the integral: $I=\int_{-7}^0\int_{-\sqrt{49-x^2}}^{\sqrt{49-x^2}}\int_{-\sqrt{49-x^2-y^2}}^0 \sqrt{x^2+y^2+z^2}\,dz\,dy\,dx$.



(a) Rewrite I in the following order of integration: $\iiint\ \ dy\,dx\,dz$.

$$I=\int_{-7}^0\int_{-\sqrt{49-z^2}}^0\int_{-\sqrt{49-x^2-z^2}}^{\sqrt{49-x^2-z^2}}\sqrt{x^2+y^2+z^2}\,dy\,dx\,dz$$

(b) Rewrite I in terms of cylindrical coordinates.

$$I=\int_{\pi/2}^{3\pi/2}\int_0^7\int_{-\sqrt{49-r^2}}^0\sqrt{r^2+z^2}\cdot r\,dz\,dr\,d\theta$$

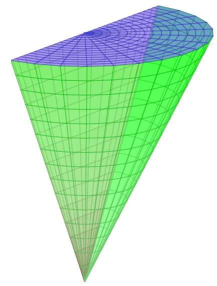
(c) Rewrite I in terms of spherical coordinates.

$$I=\int_{\pi/2}^{3\pi/2}\int_{\pi/2}^{\pi}\int_0^7\rho\cdot\rho^2\sin(\varphi)\,d\rho\,d\varphi\,d\theta$$

We have a portion of the ball $x^2+y^2+z^2\le 49$. In particular, $-\sqrt{49-x^2-y^2}\le z\le 0$ so we are dealing with the lower half (in cylindrical coordinates we have $-\sqrt{49-r^2}\le z\le 0$ and in spherical coordinates this means $\pi/2\le\varphi\le\pi$). Next, $-\sqrt{49-x^2}\le y\le\sqrt{49-x^2}$ and $-7\le x\le 0$. This is the left half of the disk centered at the origin of radius 7 (in cylindrical coordinates this says that $\pi/2\le\theta\le 3\pi/2$ and $0\le r\le 7$). Imagine a ray emanating from the origin. You start off in this region and then exit when you hit the sphere: $x^2+y^2+z^2=49$ ($\rho^2=49$). Thus $0\le\rho\le 7$.

To get the bounds for part (a), consider that we are working inside the sphere $x^2+y^2+z^2=49$. Solving for y, we get $y=\pm\sqrt{49-x^2-z^2}$. We keep both of these bounds since our original integral indicates the y-direction is not cut down. Squishing out y, we get (part of) a disk bounded by $x^2+z^2=49$. Solving for x, we get $x=\pm\sqrt{49-z^2}$. But our original integral has bounds forces $x\le 0$. Thus $-\sqrt{49-z^2}\le x\le 0$. Finally, our original integral also forces $z\le 0$, so we have $-7\le z\le 0$.

9. (14 points) Let E be the region above the cone $z = 2\sqrt{x^2+y^2}$, below $z = 18$, and where $y \geq 0$.



(a) Set up $\iiint_E x + z\,dV$ in rectangular coordinates.

(b) Set up $\iiint_E x + z\,dV$ in cylindrical coordinates.

(c) Set up $\iiint_E x + z\,dV$ in spherical coordinates.

Notice that the cone $z = 2\sqrt{x^2+y^2}$ is the bottom of our region and the horizontal plane $z = 18$ is the top. Intersecting these will help determine the bounds for x and y. We get that $2\sqrt{x^2+y^2} = 18$ and so $\sqrt{x^2+y^2} = 9$. Thus projecting out the z-direction we are left with a half-disk (because $y \geq 0$) determined by $x^2+y^2 = 9^2$. Solving for y we get $y = \pm\sqrt{81-x^2}$ and thus our y-bounds are $0 \leq y \leq \sqrt{81-x^2}$ (keeping in mind that $y \geq 0$). Finally, x should range from -9 to 9 (the radius of the disk.

$$\text{Answer (a):} \quad \int_{-9}^{9}\int_{0}^{\sqrt{81-x^2}}\int_{2\sqrt{x^2+y^2}}^{18} (x+z)\,dz\,dy\,dx$$

Alternatively, we could use $-\sqrt{81-y^2} \leq x \leq \sqrt{81-y^2}$ and $0 \leq y \leq 9$ for x and y bounds and get:

$$\text{Alternate answer (a):} \quad \int_{0}^{9}\int_{-\sqrt{81-y^2}}^{\sqrt{81-y^2}}\int_{2\sqrt{x^2+y^2}}^{18} (x+z)\,dz\,dx\,dy$$

We could also have started with x or y bounds. For example, in the y-direction, E is bounded by $y = 0$ and the cone: $z = 2\sqrt{x^2+y^2}$. Solving the cone's equation for y yields $z^2 = 4(x^2+y^2)$ so $y^2 = z^2/4 - x^2$ and thus $y = \pm\sqrt{z^2/4 - x^2}$ (where only the positive solution matters to us). Then collapsing out y, we are left with a region determined by where the cone intersects with $y = 0$: $z = 2\sqrt{x^2+0^2}$ so $z = 2|x|$ and $z = 18$. Intersecting these we have $2|x| = 18$ and again get x-bounds $-9 \leq x \leq 9$. Thus another possible answer is:

$$\text{Alternate answer (a):} \quad \int_{-9}^{9}\int_{2|x|}^{18}\int_{-\sqrt{z^2/4-x^2}}^{\sqrt{z^2/4-x^2}} (x+z)\,dy\,dx\,dz$$

Although we *could* set this up in the other 3 orders of integration, I will stop here.

Next, for cylindrical coordinates, we change the z bounds already found to our new coordinate system: $2r \leq z \leq 18$ since $\sqrt{x^2+y^2} = r$. Then we are left with a half disk in the xy-plane: $x^2+y^2 \leq 81$ and $y \geq 0$. Thus $r^2 \leq 81$ so $0 \leq r \leq 9$ and $y \geq 0$ indicates θ should sweep from the positive x-axis to the negative x-axis: $0 \leq \theta \leq \pi$. Converting $x = r\cos(\theta)$ and including the Jacobian $J = r$, we get:

$$\text{Answer (b):} \quad \int_{0}^{\pi}\int_{0}^{9}\int_{2r}^{18} (r\cos(\theta) + z)\cdot r\,dz\,dr\,d\theta$$

Finally, for spherical coordinates, consider a ray emanating from the origin. We start off in this region (so our lower ρ bound is 0) and exit when we hit the horizontal plane. Notice that $z = 18$ means $\rho\cos(\varphi) = 18$ so that $\rho = 18/\cos(\varphi) = 18\sec(\varphi)$ is our upper ρ bound. Next, for φ, image sweeping out from the positive z-axis. We start off in the region (so the lower bound for φ is 0) and exit when we sweep out to the cone: $z = 2\sqrt{x^2+y^2}$. In spherical coordinates the cone's equation is $\rho\cos(\varphi) = 2\sqrt{x^2+y^2} = 2r = 2\rho\sin(\varphi)$ so $\cos(\varphi) = 2\sin(\varphi)$ and thus $\tan(\varphi) = 1/2$. Therefore, our upper φ bound is $\varphi = \arctan(1/2)$. Alternatively we could draw a profile picture of the region E and invertigate the right triangle determined by the cone. We would have a triangle whose adjacent side is 18 units tall (determined by the top $z = 18$) and the opposite side is 9 units long (determined by the radius of our half disk top). This again tells us that the angle determining the cone is $\varphi = \arctan(9/18) = \arctan(1/2)$. Finally, we already know θ bounds from the previous part, use our spherical coordinate formulas to transform x and z, and include our Jacobian $J = \rho^2\sin(\varphi)$ to get:

$$\text{Answer (c):} \quad \int_{0}^{\pi}\int_{0}^{\arctan(1/2)}\int_{0}^{18\sec(\varphi)} (\rho\cos(\theta)\sin(\varphi) + \rho\cos(\varphi))\cdot\rho^2\sin(\varphi)\,d\rho\,d\varphi,d\theta$$