Name: Answer Key (Thanks Noah!!) **Be sure to show your work!**

1. (17 points) Vector Basics: Let $\mathbf{u} = \langle 1, 3, 2\rangle$, $\mathbf{v} = \langle 1, 2, -1\rangle$, and $\mathbf{w} = \langle 0, -1, 1\rangle$.

(a) Compute $\text{proj}_{\mathbf{v}}(\mathbf{w})$.

$$\text{proj}_{\mathbf{v}}(\mathbf{w}) = \frac{\mathbf{w} \bullet \mathbf{v}}{|\mathbf{v}|^2}\mathbf{v} = \frac{0(1) + (-1)(2) + 1(-1)}{\sqrt{1^2 + 2^2 + (-1)^2}^2}\mathbf{v} = \frac{-3}{6}\langle 1, 2, -1\rangle = \boxed{\frac{-1}{2}\langle 1, 2, -1\rangle}$$

(b) Find the area of the triangle with one side spanned by $\mathbf{u}$ and another spanned by $\mathbf{v}$.

The triangle with one side spanned by $\mathbf{u}$ and another spanned by $\mathbf{v}$ has area equal to one-half of the area of the *parallelogram* spanned by $\mathbf{u}$ and $\mathbf{v}$. This latter area is given by the length of the vector $\mathbf{u} \times \mathbf{v}$, hence, we compute $\dfrac{|\mathbf{u} \times \mathbf{v}|}{2}$ to find the desired area.

$$\begin{array}{cccccc} & \mathbf{i} & \mathbf{j} & \mathbf{k} & & \\ & 1 & 3 & 2 & = & \mathbf{u} \\ \times & 1 & 2 & -1 & = & \mathbf{v} \\ \hline & -7 & 3 & -1 & = & \mathbf{u} \times \mathbf{v} \end{array} \implies \frac{|\mathbf{u} \times \mathbf{v}|}{2} = \frac{\sqrt{(-7)^2 + 3^2 + (-1)^2}}{2} = \frac{\sqrt{59}}{2} \implies \text{Area} = \boxed{\frac{\sqrt{59}}{2}}$$

(c) Find two unit vectors which are parallel to $\mathbf{u}$.

A vector parallel to $\mathbf{u}$ must be a scalar multiple of $\mathbf{u}$; to find two *unit vectors* parallel to $\mathbf{u}$ we normalize $\mathbf{u}$ for the first and multiply by (-1) for the second:

$$|\mathbf{u}| = \sqrt{1^2 + 3^2 + 2^2} = \sqrt{14} \implies \boxed{\pm\frac{1}{\sqrt{14}}\langle 1, 3, 2\rangle}$$

(d) Find the angle between $\mathbf{v}$ and $\mathbf{w}$ (don't worry about evaluating inverse trig. functions).

$$\begin{array}{rclcl} \mathbf{v} \bullet \mathbf{w} & = & 1(0) + 2(-1) + (-1)(1) & = & -3 \\ |\mathbf{v}| & = & \sqrt{1^2 + 2^2 + (-1)^2} & = & \sqrt{6} \\ |\mathbf{w}| & = & \sqrt{0^2 + (-1)^2 + 1^2} & = & \sqrt{2} \end{array} \implies \theta = \arccos\left(\frac{-3}{\sqrt{6}\sqrt{2}}\right) = \boxed{\arccos\left(\frac{-\sqrt{3}}{2}\right)}$$

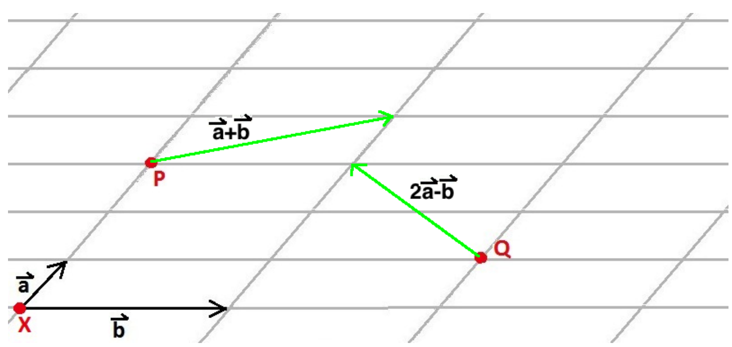
Is this angle... **right**, **acute**, or $\boxed{\textbf{obtuse}}$? (Circle your answer.) [Because $\mathbf{v} \bullet \mathbf{w} < 0$.]

(e) Match the statement on the left to the corresponding statement on the right...

$\boxed{\text{D}}$	$\mathbf{a} \times \mathbf{b} = \mathbf{0}$	**A)** is always true
$\boxed{\text{B}}$	$\mathbf{a} \bullet \mathbf{b} = 0$	**B)** $\mathbf{a}$ and $\mathbf{b}$ are orthogonal
$\boxed{\text{A}}$	$\mathbf{a} \bullet (\mathbf{b} \times \mathbf{b}) = 0$	**C)** is nonsense
$\boxed{\text{C}}$	$(\mathbf{a} \bullet \mathbf{b}) \times \mathbf{a} = \mathbf{0}$	**D)** $\mathbf{a}$ and $\mathbf{b}$ are parallel

Notice that since $\mathbf{b} \times \mathbf{b} = \mathbf{0}$, $\mathbf{a} \bullet (\mathbf{b} \times \mathbf{b}) = \mathbf{a} \bullet \mathbf{0} = 0$. On the other hand, $\mathbf{a} \bullet \mathbf{b}$ is a scalar so $(\mathbf{a} \bullet \mathbf{b}) \times \mathbf{b}$ is nonsense – you can't cross a scalar with a vector!

(f) The vectors $\mathbf{a}$ and $\mathbf{b}$ are shown to the right. They are based at the point X. Sketch the vector $\mathbf{a} + \mathbf{b}$ based at the point P and sketch the vector $2\mathbf{a} - \mathbf{b}$ based at the point Q.



2. (10 points) Let ℓ_1 be parametrized by $\mathbf{r}_1(t) = \langle 1, 1, 3\rangle + \langle 2, 1, -1\rangle t$ and let ℓ_2 be the line which passes through the points $P = (2, 0, 2)$ and $Q = (3, 4, 2)$. Determine if ℓ_1 and ℓ_2 are... (circle the correct answer)

the same, parallel (but not the same), intersecting, or $\boxed{\text{skew}}$.

First, we parameterize the line ℓ_2 by $\mathbf{r}_2(t) = P + \vec{PQ}t = \langle 2, 0, 2\rangle + \langle 1, 4, 0\rangle t$. Notice that the vectors $\mathbf{r}_1'(t) = \langle 2, 1, -1\rangle$ and $\mathbf{r}_2'(t) = \langle 1, 4, 0\rangle$ are not scalar multiples of one another which implies that ℓ_1 and ℓ_2 are not *parallel* nor, consequently, the *same* line. It remains to see if they are intersecting or skew lines.

Let's see if they intersect: $\mathbf{r}_1(t) = \mathbf{r}_2(s)$ implies that $1 + 2t = 2 + s$, $1 + t = 4s$, and $3 - t = 2$. The third equation yields $t = 1$ from with the second equation gives us $s = \dfrac{1}{2}$. Notice though, the first equation is inconsistent with these values of t and s: $1 + 2(1) \neq 2 + \left(\dfrac{1}{2}\right)$. Hence, we see ℓ_1 and ℓ_2 do *not* intersect and are instead, *skew* lines.

3. (14 points) Plane old geometry.

(a) Find the (scalar) equation of the plane through the points $A = (1, 0, 2)$, $B = (-1, 2, 1)$, and $C = (3, 1, 1)$.

To give an equation for the specified plane, we require a *normal* vector and a point within the plane. To obtain the normal vector we cross two vectors parallel to the plane (namely $\vec{AB}$ and $\vec{AC}$). We then fit the plane through one of the points (in this case, A).

$$\begin{array}{rrrrcccc} & \mathbf{i} & \mathbf{j} & \mathbf{k} & & & & \\ & -2 & 2 & -1 & = & B - A & = & \vec{AB} \\ \times & 2 & 1 & -1 & = & C - A & = & \vec{AC} \\ \hline & -1 & -4 & -6 & = & \mathbf{n} & & \end{array} \implies -1(x-1)-4(y-0)-6(z-2) = 0 \implies \boxed{x + 4y + 6z = 13}$$

(b) The planes with scalar equations $x + 2y + 3z + 4 = 0$ and $-3x + z + 5 = 0$ are... (circle the correct answer)

parallel, $\boxed{\text{perpendicular}}$, both, or neither.

Notice the first plane has normal vector $\mathbf{n}_1 = \langle 1, 2, 3 \rangle$ and the second, $\mathbf{n}_2 = \langle -3, 0, 1 \rangle$. Notice that $\mathbf{n}_1 \bullet \mathbf{n}_2 = 1(-3) + 2(0) + 3(1) = 0$, so the normal vectors, and consequently, the planes themselves, are perpendicular.

4. (8 points) Is the curvature of $y = x^3 + 3x^2 - 1$ ever zero? $\boxed{\text{Yes}}$ / No

We use the special formula for curvature of a graph in $\mathbb{R}^2$.

$$\begin{array}{c} y' = 3x^2 + 6x \\ y'' = 6x + 6 \end{array} \implies \kappa(x) = \frac{|6x + 6|}{\left(1 + (3x^2 + 6x)^2\right)^{\frac{3}{2}}} = 0 \qquad \text{if } x = -1$$

5. (14 points) Parameterization, arc length, and a line integral.

(a) Let C be the upper-half of the circle $x^2 + y^2 = 9$. Parameterize C and then find its arc length using an integral. [You must compute an integral.]

Noting that C is a portion of a circle, we use polar coordinates (*i.e.*, $x = r\cos(t)$, $y = r\sin(t)$) to parameterize the curve. Note that we restrict t to the interval $[0, \pi]$ since we only want the upper-half of the circle. Also, the *arc length formula* is simply the line integral of 1 along C.

$$C : \mathbf{r}(t) = \langle 3\cos(t), 3\sin(t) \rangle, \quad 0 \le t \le \pi \qquad \text{and so} \qquad |\mathbf{r}'(t)| = \left|\langle -3\sin(t), 3\cos(t) \rangle\right| = \sqrt{9\sin^2(t) + 9\cos^2(t)} = 3$$

$$\text{Arc Length} = \int_C 1\, ds = \int_0^\pi |\mathbf{r}'(t)|\, dt = \int_0^\pi 3\, ds = \boxed{3\pi}$$

(b) Find $\int_C (x + y)\, ds$. Here we just plug in what we found in the previous part.

$$\int_C (x + y)\, ds = \int_0^\pi (3\cos(t) + 3\sin(t))3\, dt = 3\left[3\sin(t) - 3\cos(t)\right]\Big|_0^\pi = 3(0 + 3(2)) = \boxed{18}$$

6. (13 points) Let C be parameterized by $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$ where $-1 \le t \le 5$.

(a) Set up the line integral $\int_C (y^2 + e^x \sin(yz))\, ds$. [Do not try to evaluate this integral. It will only end in tears.]

$$\mathbf{r}'(t) = \langle 1, 2t, 3t^2 \rangle \implies |\mathbf{r}'(t)| = \sqrt{9t^4 + 4t^2 + 1} \qquad \int_C (y^2 + e^x \sin(yz))\, ds = \boxed{\int_{-1}^5 \left(t^4 + e^t \sin\left(t^5\right)\right) \sqrt{9t^4 + 4t^2 + 1}\, dt}$$

(b) Find the curvature of $\mathbf{r}(t)$.

$$\begin{array}{rrrrcc} & \mathbf{i} & \mathbf{j} & \mathbf{k} & & \\ & 1 & 2t & 3t^2 & = & \mathbf{r}'(t) \\ \times & 0 & 2 & 6t & = & \mathbf{r}''(t) \\ \hline & 6t^2 & -6t & 2 & = & \mathbf{r}'(t) \times \mathbf{r}''(t) \end{array} \implies \kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|} = \frac{\sqrt{36t^4 + 36t^2 + 4}}{\left(9t^4 + 4t^2 + 1\right)^{\frac{3}{2}}} = \boxed{\frac{2\sqrt{9t^4 + 9t^2 + 1}}{\left(9t^4 + 4t^2 + 1\right)^{\frac{3}{2}}}}$$

7. (12 points) Find the TNB-frame for $\mathbf{r}(t) = \langle \sqrt{2}\cos(t), \sqrt{2}\cos(t), 2\sin(t)\rangle$.

Recall that $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$, $\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}$ and $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$.

$$\mathbf{r}'(t) = \langle -\sqrt{2}\sin(t), -\sqrt{2}\sin(t), 2\cos(t)\rangle \quad \Longrightarrow \quad |\mathbf{r}'(t)| = \sqrt{2\sin^2(t) + 2\sin^2(t) + 4\cos^2(t)} = 2$$

$$\Longrightarrow \quad \mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \boxed{\left\langle -\frac{1}{\sqrt{2}}\sin(t), -\frac{1}{\sqrt{2}}\sin(t), \cos(t)\right\rangle}$$

$$\mathbf{T}'(t) = \left\langle -\frac{1}{\sqrt{2}}\cos(t), -\frac{1}{\sqrt{2}}\cos(t), -\sin(t)\right\rangle \quad \Longrightarrow \quad |\mathbf{T}'(t)| = \sqrt{\frac{1}{2}\cos^2(t) + \frac{1}{2}\cos^2(t) + \sin^2(t)} = 1$$

$$\Longrightarrow \quad \mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \boxed{\left\langle -\frac{1}{\sqrt{2}}\cos(t), -\frac{1}{\sqrt{2}}\cos(t), -\sin(t)\right\rangle}$$

$$\begin{array}{cccccc} & \mathbf{i} & \mathbf{j} & \mathbf{k} & & \\ & -\frac{1}{\sqrt{2}}\sin(t) & -\frac{1}{\sqrt{2}}\sin(t) & \cos(t) & = & \mathbf{T}(t) \\ \times & -\frac{1}{\sqrt{2}}\cos(t) & -\frac{1}{\sqrt{2}}\cos(t) & -\sin(t) & = & \mathbf{N}(t) \\ \hline & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & = & \mathbf{T}(t) \times \mathbf{N}(t) \end{array} \quad \Longrightarrow \quad \mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t) = \boxed{\left\langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right\rangle}$$

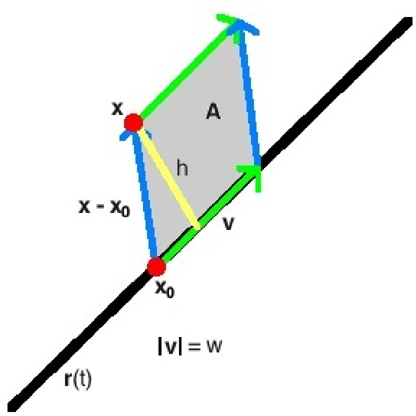
Does this curve lie in a plane? Why or why not?

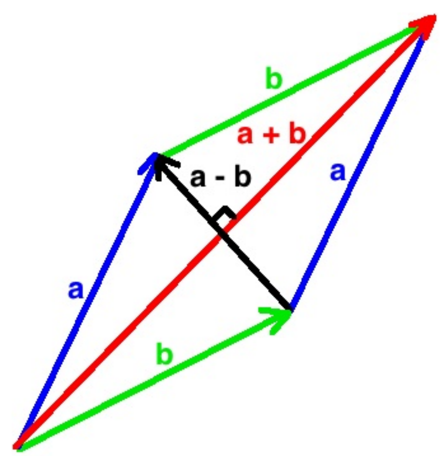
$\boxed{\text{Yes. } \mathbf{B}(t) \text{ is constant}}$, so our curve lies in its osculating plane. In fact, this is a circle (radius 2) lying in the plane $x = y$.

8. (12 points) No numbers here. Choose **ONE** of the following:

I. Let ℓ be a line parameterized by $\mathbf{r}(t) = \mathbf{x}_0 + \mathbf{v}t$. Let $\mathbf{x}$ be some point (not necessarily on this line). Explain why $\dfrac{|\mathbf{v} \times (\mathbf{x} - \mathbf{x}_0)|}{|\mathbf{v}|}$ is the distance from $\mathbf{x}$ to the line ℓ. [Your explanation should include a picture.]

$|\mathbf{v} \times (\mathbf{x} - \mathbf{x}_0)|$ is the area of the parallelogram spanned by $\mathbf{v}$ and $\mathbf{x} - \mathbf{x}_0$. This parallelogram has width $w = |\mathbf{v}|$ and height h. This height, h is precisely the distance from ℓ (parameterized by $\mathbf{r}(t)$) to $\mathbf{x}$. Therefore, we have that $\dfrac{|\mathbf{v} \times (\mathbf{x} - \mathbf{x}_0)|}{|\mathbf{v}|} = \dfrac{A}{w} = \dfrac{wh}{w} = h$, the distance from ℓ to $\mathbf{x}$. [See the figure below and to the left.]





II. Suppose $\mathbf{a}$ and $\mathbf{b}$ have the same length. Show that $\mathbf{a} + \mathbf{b}$ and $\mathbf{a} - \mathbf{b}$ are orthogonal.
[Your explanation should include a picture.]

We check that $(\mathbf{a} + \mathbf{b}) \bullet (\mathbf{a} - \mathbf{b}) = 0$. Let $c = |\mathbf{a}| = |\mathbf{b}|$ be the length of these vectors. Then...

$$(\mathbf{a} + \mathbf{b}) \bullet (\mathbf{a} - \mathbf{b}) = \mathbf{a} \bullet \mathbf{a} + \mathbf{b} \bullet \mathbf{a} - \mathbf{a} \bullet \mathbf{b} - \mathbf{b} \bullet \mathbf{b} = \mathbf{a} \bullet \mathbf{a} - \mathbf{b} \bullet \mathbf{b} = |\mathbf{a}|^2 - |\mathbf{b}|^2 = c^2 - c^2 = 0.$$

Hence, $\mathbf{a} + \mathbf{b}$ and $\mathbf{a} - \mathbf{b}$ are orthogonal. [See the figure above and to the right.]