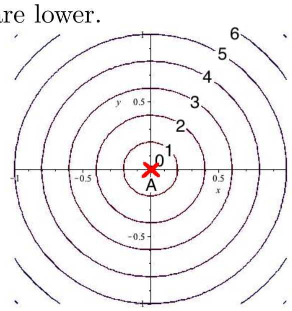
Math 2130-101 # Test #2 **July 22^{nd}, 2014**

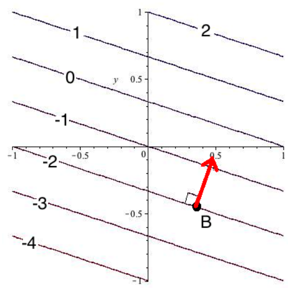
Name: Answer Key **Be sure to show your work!**

If $F(x,y)=C$, then $\dfrac{dy}{dx}=-\dfrac{F_x}{F_y}$ If $F(x,y,z)=C$, then $\dfrac{\partial z}{\partial x}=-\dfrac{F_x}{F_z}$ and $\dfrac{\partial z}{\partial y}=-\dfrac{F_y}{F_z}$

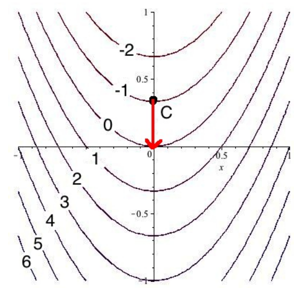
1. (11 points) Three level curve plots are shown below. I have labeled the levels so you know which curves are higher and which are lower.



$\boxed{5\sqrt{x^2+y^2}}$



$\boxed{x+3y+1}$



$\boxed{4x^2-3y}$

(a) The plots above correspond to 3 of the functions listed here: $f(x,y)=9-x^2-y^2$, $f(x,y)=x+3y-1$, $f(x,y)=5\sqrt{x^2+y^2}$, $f(x,y)=4x^2-3y$, and $f(x,y)=4y^2-3x$. Write the correct formula below each plot.

(b) Sketch a gradient vector at the points A, B, and C. If the vector is $\mathbf{0}$, draw an "X" on the point. [Don't worry about having the correct length. I'm just looking for the correct direction.]

(c) If A, B, or C is a critical point, write what kind of point it is (i.e. min, max, saddle, or other).

Notice that though $f(x,y)=9-x^2-y^2$ has circles for level curves ($c=9-x^2-y^2$), its level curve at $c=0$ is a circle of radius 3 ($0=9-x^2-y^2$ so $x^2+y^2=3^2$); this does not appear on the graphs above, thus its level curve plot is not included. $f(x,y)=x+3y-1$ is the only function appearing with level curves which are lines ($c=x+3y-1$) so it is the second graph. $f(x,y)=5\sqrt{x^2+y^2}$ has level curves that are circles of radius $\sqrt{c/5}$ ($(c/5)^2=x^2+y^2$); we see this in the first graph. $f(x,y)=4x^2-3y$ has level curves which are parabolas opening upwards ($c=4x^2-3y$). This differs from $f(x,y)=4y^2-3x$ which has level curves that are parabolas opening to the right ($c=4y^2-3x$). Thus $f(x,y)=4x^2-3y$ is the function whose level curves are exhibited in the third plot.

Only the point A is a critical point, and as it is the lowest level curve, we conclude it a $\boxed{\textit{minimum}}$. In fact, this isn't just a relative minimum for the cone, it's a global minimum!

2. (8 points) Let $z=f(x,y)$ where $x=r\cos(\theta)$ and $y=r\sin(\theta)$. Show that $\left(\dfrac{\partial z}{\partial x}\right)^2+\left(\dfrac{\partial z}{\partial y}\right)^2=\left(\dfrac{\partial z}{\partial r}\right)^2+\dfrac{1}{r^2}\left(\dfrac{\partial z}{\partial \theta}\right)^2$.

By the chain rule...

$$\frac{\partial z}{\partial r}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial r}=\frac{\partial z}{\partial x}\cos(\theta)+\frac{\partial z}{\partial y}\sin(\theta)\quad\text{and}\quad\frac{\partial z}{\partial \theta}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta}=\frac{\partial z}{\partial x}(-r\sin(\theta))+\frac{\partial z}{\partial y}(r\cos(\theta)).$$

We begin with the right-hand side of the equality and derive the left-hand side:

$$\begin{aligned}
\left(\frac{\partial z}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2 &= \left(\frac{\partial z}{\partial x}\cos(\theta)+\frac{\partial z}{\partial y}\sin(\theta)\right)^2+\frac{1}{r^2}\left(\frac{\partial z}{\partial x}(-r\sin(\theta))+\frac{\partial z}{\partial y}(r\cos(\theta))\right)^2\\
&= \left(\frac{\partial z}{\partial x}\cos(\theta)+\frac{\partial z}{\partial y}\sin(\theta)\right)^2+\frac{1}{r^2}(r^2)\left(\frac{\partial z}{\partial x}(-\sin(\theta))+\frac{\partial z}{\partial y}(\cos(\theta))\right)^2\\
&= \left(\frac{\partial z}{\partial x}\right)^2\cos^2(\theta)+2\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}\sin(\theta)\cos(\theta)+\left(\frac{\partial z}{\partial y}\right)^2\sin^2(\theta)\\
&\qquad+\left(\frac{\partial z}{\partial x}\right)^2\sin^2(\theta)-2\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}\sin(\theta)\cos(\theta)+\left(\frac{\partial z}{\partial y}\right)^2\sin^2(\theta)\\
&= \left(\frac{\partial z}{\partial x}\right)^2\left(\sin^2(\theta)+\cos^2(\theta)\right)+\left(\frac{\partial z}{\partial y}\right)^2\left(\sin^2(\theta)+\cos^2(\theta)\right) \quad = \quad \left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2.
\end{aligned}$$

3. (10 points) Let $x^3y^2 + e^{xyz} + z^2 = 2$.

(a) Find an equation for the plane tangent to the above surface at the point $(x, y, z) = (0, 1, -1)$.

Notice here that we have a level surface of $F(x, y, z) = x^3y^2 + e^{xyz} + z^2$ where $F(x, y, z) = 2$. Also, note that the point does in fact lie upon this surface, that is, $F(0, 1, -1) = (0)^3(1)^2 + e^{(0)(1)(-1)} + (-1)^2 = 0 + e^0 + 1 = 2$. To find an equation for the tangent plane we need to find a normal vector. This is given by the gradient of F evaluated at the given point.

$$\nabla F(x, y, z) = \langle 3x^2y^2 + yze^{xyz}, 2x^3y + xze^{xyz}, xye^{xyz} + 2z\rangle$$

$$\nabla F(0, 1, -1) = \langle 0 + (1)(-1)e^0, 0 + 0(-1)e^0, 0(1)e^0 + 2(-1)\rangle = \langle -1, 0, -2\rangle$$

$$-1(x - 0) + 0(y - 1) + (-2)(z - (-1)) = 0 \qquad \Longrightarrow \qquad \boxed{x + 2z + 2 = 0}$$

(b) Considering z as a variable depending on x and y (defined implicitly above), find $\dfrac{\partial z}{\partial x}$.

With $F(x, y, z) = C$, we have that $\dfrac{\partial z}{\partial x} = -\dfrac{F_x}{F_z}$. Conveniently, we already found the necessary partial derivatives when we computed ∇F.

$$\boxed{\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{3x^2y^2 + yze^{xyz}}{xye^{xyz} + 2z}}$$

4. (10 points) Limits

(a) Show the following limit **does** exist: $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 + xy + y^2}{\sqrt{x^2 + y^2}}$

To show this limit exists, we utilize polar coordinates (which is about the only tool we have discussed to show the existence of a limit). Let $x = r\cos(\theta)$ and $y = r\sin(\theta)$. Then $x^2 + y^2 = r^2$. Recall that the origin has polar coordinates $(r, \theta) = (0, \theta)$ no matter what θ is.

$$\lim_{(x,y)\to(0,0)} \frac{x^2 + xy + y^2}{\sqrt{x^2 + y^2}} = \lim_{(r,\theta)\to(0,\theta)} \frac{r^2 + r^2\cos(\theta)\sin(\theta)}{r} = \lim_{(r,\theta)\to(0,\theta)} \frac{r^2(1 + \cos(\theta)\sin(\theta))}{r} = \lim_{(r,\theta)\to(0,\theta)} r(1+\cos(\theta)\sin(\theta)) = 0$$

Note: The final expression involves θ. So it is important to notice that $1 + \cos(\theta)\sin(\theta)$ is bounded (by 0 and 2). Thus $0 \le r(1 + \cos(\theta)\sin(\theta)) \le 2r$, so as r goes to 0, so does $r(1 + \cos(\theta)\sin(\theta)$.

(b) Show the following limit **does not** exist: $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 + xy}{2x^2 + y^2}$

To show this limit does not exist we must approach the origin along two different curves and show the limit converges to differing values. Notice that along $x = 0$ we get:

$\displaystyle\lim_{(0,y)\to(0,0)} \frac{(0)^2 + (0)y}{2(0)^2 + y^2} = \lim_{y\to 0} \frac{0}{y^2} = 0.$ However, along $y = 0$ we get: $\displaystyle\lim_{(x,0)\to(0,0)} \frac{x^2 + x(0)}{2x^2 + (0)^2} = \lim_{x\to 0} \frac{x^2}{2x^2} = \lim_{x\to 0} \frac{1}{2} = \frac{1}{2}.$

Since $0 \ne \frac{1}{2}$ (i.e. the expression heads to different values along different curves), the limit does not exist.

5. (9 points) Suppose we have a function of two variables: $f(x, y)$.

(a) Suppose that $f_{xy}(1, 2) = 3$ and $f_{yx}(1, 2) = 4$. What (if anything) can I conclude about f or its partials?

Either f_{xy} or f_{yx} must not be continuous at the point $(1, 2)$ since if both were continuous, *Clairaut's Theorem* would guarantee that they are equal.

(b) Suppose f_x and f_y exist everywhere. Can I conclude f is a continuous function? **YES** / $\boxed{\textbf{NO}}$

Although this typically isn't a problem for us, it is possible to find functions whose partial derivatives exist everywhere yet fail to be differentiable at certain points.

(c) Suppose f_x and f_y are continuous everywhere. Can I conclude that f is differentiable? $\boxed{\textbf{YES}}$ / **NO**

Recall that the first partials, f_x and f_y, being continuous implies that f is differentiable. This implies that both the first partials, f_x and f_y, exist and that f is continuous. The converses of these statements do not, however, hold in general.

6. (12 points) Let $f(x,y) = -x^4 + 4xy - 2y^2 - 3$.

(a) Find the gradient of f and the Hessian matrix of f.

$$\nabla f = \langle f_x, f_y \rangle = \langle -4x^3 + 4y, 4x - 4y \rangle \qquad \text{and} \qquad H_f = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} -12x^2 & 4 \\ 4 & -4 \end{bmatrix}$$

(b) Find the quadratic approximation of f at $(x,y) = (0,-1)$.

Note first that $f(0,-1) = -5$, $\nabla f(0,-1) = \langle -4, 4 \rangle$ and $H_f(0,-1) = \begin{bmatrix} 0 & 4 \\ 4 & -4 \end{bmatrix}$ and recall that the quadratic approximation of f at $(x,y) = (a,b)$ is

$$Q(x,y) = f(a,b) + \nabla f(a,b) \bullet \langle x-a, y-b \rangle + \frac{1}{2} \begin{bmatrix} x-a & y-b \end{bmatrix} H_f(a,b) \begin{bmatrix} x-a \\ y-b \end{bmatrix}.$$

Thus the quadratic approximation of f at $(x,y) = (0,1)$ is...

$$\begin{aligned} Q(x,y) &= -5 + \langle -4, 4 \rangle \bullet \langle x, y+1 \rangle + \tfrac{1}{2} \begin{bmatrix} x & y+1 \end{bmatrix} \begin{bmatrix} 0 & 4 \\ 4 & -4 \end{bmatrix} \begin{bmatrix} x \\ y+1 \end{bmatrix} \\ &= \boxed{-5 - 4x + 4(y+1) + 4x(y+1) - 2(y+1)^2} \end{aligned}$$

(c) Find an classify the critical point(s) of $f(x,y)$.
[Use the "2^{nd}-derivative" test to determine if critical points are relative max's, min's or saddle points.]

Critical points are points (a,b) such that $\nabla f(a,b) = \mathbf{0}$ or does not exist. Since ∇f exists everywhere, we just need to solve $\nabla f(x,y) = \langle 0, 0 \rangle$.

$$\begin{matrix} -4x^3 + 4y = 0 \\ 4x - 4y = 0 \end{matrix} \implies \begin{matrix} y = x^3 \\ x = y \end{matrix} \implies y = x \text{ and } x^3 - x = 0 \implies y = x \text{ and } x(x-1)(x+1) = 0 \implies y = x = 0, \pm 1.$$

Thus we have three critical points to check using our "2^{nd}-derivative" test: $(0,0)$, $(1,1)$ and $(-1,-1)$. Plugging these into the Hessian we see...

$$H_f(0,0) = \begin{bmatrix} 0 & 4 \\ 4 & -4 \end{bmatrix} \implies \det(H_f(0,0)) = -16 < 0 \implies \text{saddle point}$$

$$H_f(1,1) = \begin{bmatrix} -12 & 4 \\ 4 & -4 \end{bmatrix} \implies \det(H_f(1,1)) = 48 - 16 = 32 > 0 \text{ and } f_{xx}(1,1) = -12 < 0 \implies \text{relative maximum}$$

$$H_f(-1,-1) = \begin{bmatrix} -12 & 4 \\ 4 & -4 \end{bmatrix} \implies \det(H_f(-1,-1)) = 48 - 16 = 32 > 0 \text{ and } f_{xx}(1,1) = -12 < 0 \implies \text{relative maximum}$$

Answer: $\boxed{(0,0) \text{ is a saddle point}}$. Both $\boxed{(1,1) \text{ and } (-1,-1) \text{ are relative minimums}}$.

7. (8 points) Let $z = \dfrac{y}{x^2}$. Use a differential (i.e. total derivative) to estimate the maximal change in z if we start at $(x,y) = (-1,2)$ and then x changes by ± 0.2 and y changes by ± 0.1.

We have $z = yx^{-2}$ so that $dz = z_x dx + z_y dy = -2yx^{-3}\,dx + x^{-2}\,dy$ estimates the actual change in z (i.e. Δz). Now we just plug in our point $(x,y) = (-1,2)$ and notice that $|dx| \leq 0.2$ and $|dy| \leq 0.1$.

$$|dz| = |-2(2)(-1)^{-3}dx + (-1)^2 dy| = |4dx + dy| \leq 4|dx| + |dy| \leq 4(0.2) + (0.1) = \boxed{0.9}$$

8. (10 points) A Directed Problem. [Assume that the function $g(x,y)$ in parts (b) and (c) is differentiable.]

(a) Let $f(x,y,z) = e^{xy^2} + y^2 z$. Find the directional derivative of f at the point $(x,y) = (0,-1,3)$ and in the same direction as $\mathbf{v} = \langle -2, 1, 3 \rangle$.

Recall that the directional derivative is only defined for unit vectors. Thus we need to normalize $\mathbf{v}$ first: $\mathbf{u} = \dfrac{\mathbf{v}}{|\mathbf{v}|} = \frac{1}{\sqrt{14}} \langle -2, 1, 3 \rangle$. Next, we find and evaluate the gradient of f...

$$\nabla f(x,y,z) = \langle y^2 e^{xy^2}, 2xye^{xy^2} + 2yz, y^2 \rangle \qquad \nabla f(0,-1,3) = \langle (-1)^2 e^{(0)(-1)^2}, 0 + 2(-1)(3), (-1)^2 \rangle = \langle 1, -6, 1 \rangle$$

Therefore, $\quad D_{\mathbf{u}} f(0,-1,3) = \nabla f(0,-1,3) \bullet \dfrac{1}{\sqrt{14}} \langle -2, 1, 3 \rangle = \langle 1, -6, 1 \rangle \bullet \dfrac{1}{\sqrt{14}} \langle -2, 1, 3 \rangle = \boxed{\dfrac{-5}{\sqrt{14}}}.$

(b) Suppose that $\nabla g(1,2) = \langle 3,4\rangle$. What is the maximum possible value of $D_{\mathbf{u}}g(1,2)$? Give a unit vector which causes this maximum to occur.

Recall, that moving in the gradient correction causes maximum positive change and this maximal change is precisely the length of the gradient. Thus, the maximum value the directional derivative can achieve (at the point $(1,2)$) is $|\nabla g(1,2)| = \sqrt{3^2+4^2} = \boxed{5}$. This value is obtained when $\mathbf{u} = \dfrac{\nabla g(1,2)}{|\nabla g(1,2)|} = \left\langle \dfrac{3}{5}, \dfrac{4}{5}\right\rangle$.

(c) Again, suppose $\nabla g(1,2) = \langle 3,4\rangle$. Mr. Pete claims that for some unit vector $\mathbf{u}$ he found that $D_{\mathbf{u}}g(1,2) = -10$. Why must Mr. Pete be wrong?

The minimal value $D_{\mathbf{u}}g(1,2)$ can obtain is $-|\nabla g(1,2)| = -5$. Since $-10 < -5$ (the minimal value) the directional derivative at $(1,2)$ cannot take on this value. Sorry, Mr. Pete, -3 points for you.

9. (10 points) Suppose $f(x,y)$ is a "nice" function (with continuous partials of all orders).

(a) $Q(x,y) = 1 + (x+3)^2 + 4(x+3)(y-5) + 4(y-5)^2$ is the quadratic approx. at $(x,y) = (-3,5)$.

$$\nabla f(-3,5) = \langle 0,0\rangle \qquad \text{and} \qquad H_f(-3,5) = \begin{bmatrix} 2 & 4 \\ 4 & 8 \end{bmatrix}$$

Be careful! Remember how the quadratic approximation is defined. $f_x(-3,5)$ is exactly the coefficient of $(x+3)$ (which does occurs so $f_x(-3,5) = 0$. Likewise, $f_y(-3,5) = 0$ (the coefficient of $y-5$). The quadratic terms are a bit trickier. They are scaled by $\frac{1}{2}$ so we must consider this when determining f_{xx} and f_{yy}. Thus, $f_{xx} = 2\cdot 1 = 2$ and $f_{yy} = 2\cdot 4 = 8$. However, with regard to the mixed partials, they are equal since we assumed continuous partials of all orders (*Claraiut's Theorem*) so we do not need to scale the value by 2. $f_{xy}(-3,5) = f_{yx}(-3,5) = \frac{1}{2}f_{xy}(-3,5) + \frac{1}{2}{}_{yx}(-3,5) = 4$.

Is $(x,y) = (-3,5)$ a critical point of $f(x,y)$? $\boxed{\textbf{YES}}$ / **NO**

If not, why not? If so, what kind (relative min, relative max, saddle point or not enough information)?

We see that $\nabla f(-3,5) = \mathbf{0}$ and conclude $(-3,5)$ a critical point. However, since $\det(H_f(-3,5)) = 2(8) - 4^2 = 0$, we $\boxed{\text{do not have enough information}}$ to classify this critical point via our "2^{nd}-derivative test."

(b) $Q(x,y) = 2x - 3x^2 + 4x(y-5) - 6(y-5)^2$ is the quadratic approx. at $(x,y) = (0,5)$.

$$\nabla f(0,5) = \langle 2,0\rangle \qquad \text{and} \qquad H_f(0,5) = \begin{bmatrix} -6 & 4 \\ 4 & -12 \end{bmatrix}$$

Is $(x,y) = (0,5)$ a critical point of $f(x,y)$? **YES** / $\boxed{\textbf{NO}}$

If not, why not? If so, what kind (relative min, relative max, saddle point or not enough information)?

We have that $(0,5)$ is not a critical point because $\nabla f(0,5) \neq \langle 0,0\rangle$.

10. (12 points) Use the method of Lagrange multipliers to find the minimum and maximum values of
$f(x,y,z) = xyz$ constrained to $x^2+2y^2+z^2 = 12$.

$\nabla f = \langle yz, xz, xy\rangle$ and $\nabla g = \langle 2x, 4y, 2z\rangle$. Our Lagrange multiplier equations are: $\nabla f = \lambda \nabla g$ and the constraint, that is,

$$yz = 2x\lambda, \qquad xz = 4y\lambda, \qquad xy = 2z\lambda, \qquad \text{and} \qquad x^2+2y^2+z^2 = 12.$$

We should symmetrize these equations. Multiply the first by x, the second by y, and the third by z. This yields: $xyz = 2x^2\lambda = 4y^2\lambda = 2z^2\lambda$. Now $\lambda = 0$ forces $\nabla f = \lambda\nabla g = 0\nabla g = \mathbf{0}$. This would force one of x, y, or z to be 0 which gives $f(x,y,z) = 0$ which is clearly not a min or max value of f. So we assume $\lambda \neq 0$ and divide. Therefore, $\dfrac{xyz}{2\lambda} = x^2 = 2y^2 = z^2$. So, we find that

$$x^2+x^2+x^2 = 12 \implies x^2 = 4 \implies x = \pm 2$$
$$2y^2 = x^2 = 4 \implies y^2 = 2 \implies y = \pm\sqrt{2}$$
$$z^2 = x^2 = 4 \implies z = \pm 2$$

We are only concerned with the product (i.e. $f(x,y,z) = xyz$) of the coordinates of our critical points. Notice, that plugging in four of these points will result in the positive answer $2\cdot 2\cdot\sqrt{2} = 4\sqrt{2}$ (ex: $f(2,-\sqrt{2},-2) = 4\sqrt{2}$) while the other four will result in the corresponding negative answer $-2\cdot 2\cdot\sqrt{2} = -4\sqrt{2}$ (ex: $f(-2,-\sqrt{2},-\sqrt{2}) = -4\sqrt{2}$). This leads us to conclude that the maximum and minimum values of f constrained to $x^2+2y^2+z^2 = 12$ are $\boxed{\pm 4\sqrt{2}}$.