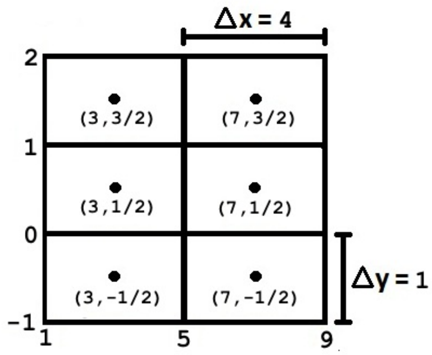
Math 2130-101 **Test #3** **July 29th, 2014**

Name: Answer Key **Be sure to show your work!**

1. (10 points) Use a double Riemann sum to approximate $\iint_R x\sin(y^2)\,dA$ where $R = [1,9]\times[-1,2]$.

Use midpoint rule and a 2×3 grid of rectangles (2 across and 3 up) to partition R. (Don't worry about simplifying.)



$$\iint_R x\sin(y^2)\,dA \approx 4\cdot 1\Big[3\sin\left(\left(\frac{-1}{2}\right)^2\right)+3\sin\left(\left(\frac{1}{2}\right)^2\right)+3\sin\left(\left(\frac{3}{2}\right)^2\right)$$
$$+7\sin\left(\left(\frac{-1}{2}\right)^2\right)+7\sin\left(\left(\frac{1}{2}\right)^2\right)+7\sin\left(\left(\frac{3}{2}\right)^2\right)\Big]$$

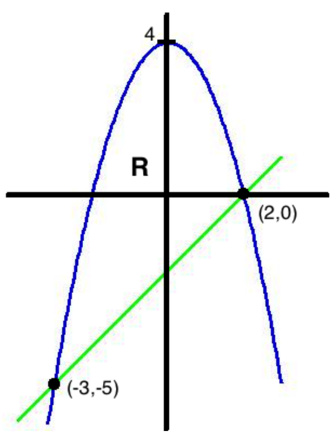
2. (10 points) Let R be the region bounded by $y = x-2$ and $y = 4-x^2$.

[Warning: One of the integrals below will have to be split into 2 pieces.]

(a) Sketch the region R.

(b) Set up the integral $\iint_R x\,dA$ using the order of integration "$dy\,dx$". Don't evaluate the integral.

(c) Set up the integral $\iint_R x\,dA$ using the order of integration "$dx\,dy$". Don't evaluate the integral.



(a) Setting our bounds equal, $x-2 = 4-x^2 \implies x^2+x-6 = 0 \implies (x+3)(x-2) = 0$, we see that these curves intersect when $x=-3$ and 2, and as a result, when $y=-5$ and 0.

(b) The bounds for the order of integration $dy\,dx$ are rather trivial, as y simply ranges from the bottom up to the top curve (i.e. $x-2\le y\le 4-x^2$) and x ranges from -3 to 2.

$$\boxed{\int_{-3}^{2}\int_{x-2}^{4-x^2} x\,dy\,dx.}$$

(c) These bounds are slightly trickier. We solve for x: $y = 4-x^2 \implies x^2 = 4-y \implies x = \pm\sqrt{4-x}$ and $y = x-2 \implies x = y+2$. Note that when $y\ge 0$, x ranges from the negative side of the parabola to the positive side of the parabola so we have $-\sqrt{4-y}\le x\le\sqrt{4-y}$. This occurs as y ranges from 0 to 4. When $y\le 0$, x ranges from the negative side of the parabola to the line $x = y+2$, so we have $-\sqrt{4-y}\le x\le y+2$ as y ranges from -5 to 0. Thus, breaking up our integral we have

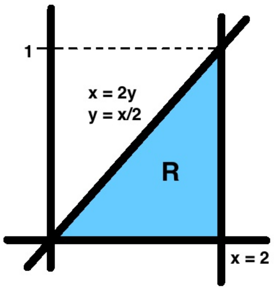
$$\iint_R x\,dx\,dy = \boxed{\int_{-5}^{0}\int_{-\sqrt{4-y}}^{y+2} x\,dx\,dy + \int_{0}^{4}\int_{-\sqrt{4-y}}^{\sqrt{4-y}} x\,dx\,dy.}$$

3. (10 points) Compute $\int_0^1\int_{2y}^{2} e^{-x^2}\,dx\,dy$. *Hint:* You cannot integrate $\int e^{-x^2}\,dx$ in terms of elementary functions.

As we cannot compute this integral in its current form, that is, with respect to x first, we must change the order of integration. To do this, we find bounds for the order $dy\,dx$.

First, our current bounds tell us that $2y\le x\le 2$ (left to right) and $0\le y\le 1$ (bottom to top). So we have a triangular region whose left side is the line $2y = x$ (which is $y = x/2$) and right side is the vertical line $x = 2$. Now we can sketch our region.

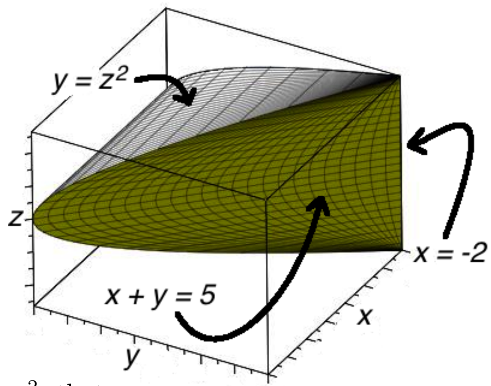
Next, from the sketch on the next page, that y ranges from 0 to the line $y = x/2$. Also, we see that x ranges from 0 to 2. Armed with this, we can proceed with our computation.



$$\int_0^1 \int_{2y}^2 e^{-x^2}\,dx\,dy = \int_0^2 \int_0^{x/2} e^{-x^2}\,dy\,dx = \int_0^2 ye^{-x^2}\Big|_0^{x/2}\,dx$$

$$= \int_0^2 \frac{x}{2}e^{-x^2}\,dx = \frac{-1}{4}e^{-x^2}\Big|_0^2 = \boxed{\frac{1}{4}\left(1-e^{-4}\right)}$$

4. (12 points) Let E be the region bounded by $y = z^2$, $x + y = 5$, and $x = -2$. A graph of this region is ever so kindly provided to the right. Set up integrals which compute the volume of E using the following orders of integration: [Do not evaluate these integrals.]



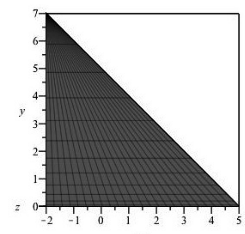
(a) $\int_?^? \int_?^? \int_?^? ???\,dz\,dy\,dx$

(b) $\int_?^? \int_?^? \int_?^? ???\,dx\,dz\,dy$

(c) $\int_?^? \int_?^? \int_?^? ???\,dy\,dx\,dz$

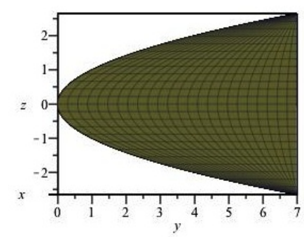
(a) With respect to this region, z is bounded above and below by the surface $y = z^2$, that is $z = \pm\sqrt{y}$ so $-\sqrt{y} \le z \le \sqrt{y}$. Now, with z's direction integrated out, we see that y is bounded above by the plane $x + y = 5$ and below by 0 (since $y = z^2$, $y \ge 0$), thus $0 \le y \le 5 - x$. With only x left, we know that x is bounded below by $x = -2$ and at the point where $x + y = 5$ crosses the x-axis, namely when $x = 5$ and $y = 0$. Hence, $-2 \le x \le 5$.

$$\boxed{\int_{-2}^5 \int_0^{5-x} \int_{-\sqrt{y}}^{\sqrt{y}} 1\,dz\,dy\,dx.}$$



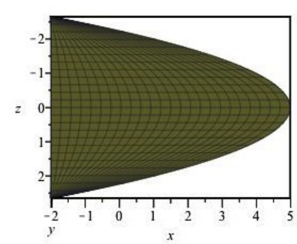
(b) Here x is bounded below by $x = -2$ (the back of our region) and above by $x + y = 5$ (the front of our region), hence $-2 \le x \le y - 5$. Once again, z is bounded by $y = z^2$, so $-\sqrt{y} \le z \le \sqrt{y}$. Here though, y is bounded below by 0 (as above), but is bounded above where the surfaces $y = z^2$, $x + y = 5$ and $x = -2$ intersect. This is where $-2 + y = 5$ or when $y = 7$ so $0 \le y \le 7$.

$$\boxed{\int_0^7 \int_{-\sqrt{y}}^{\sqrt{y}} \int_{-2}^{5-y} 1\,dx\,dz\,dy.}$$

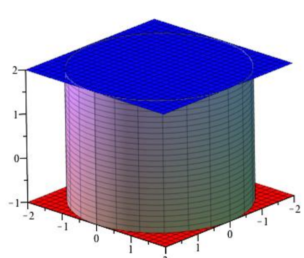


(c) Here y is bounded below by $y = z^2$ (the left side of our region) and above by the plane $x + y = 5$ (the right side), so, simply enough, we have $z^2 \le y \le 5 - x$. Again, we have that x is bounded below by -2 but is now bounded above by the intersection of $y = z^2$ and $x + y = 5$, that is the curve $x + z^2 = 5$, hence, $-2 \le x \le 5 - z^2$. Here z is bounded by the intersection of $x + z^2 = 5$ and $x = -2$, where $-2 + z^2 = 5$ or $z^2 = 7$ so we have $-\sqrt{7} \le z \le \sqrt{7}$.

$$\boxed{\int_{-\sqrt{7}}^{\sqrt{7}} \int_2^{5-z^2} \int_{z^2}^{5-x} 1\,dy\,dx\,dz.}$$



5. (10 points) Compute $\iiint_E x^2 + y^2\,dV$ where E is bounded by $z = -1$, $z = 2$, and $x^2 + y^2 = 4$.



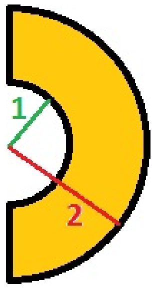
The region bounded by these three functions is a cylinder, this (along with the appearance of x^2+y^2) serves as motivation to change to cylindrical coordinates. Keep in mind $x^2+y^2 = r^2$ and don't forget the Jabobian: $J = r$. In cylindrical coordinates, we have that E's boundary is $z = -1$, $z = 2$, and $r^2 = 4$. So $-1 \le z \le 2$ and $0 \le r \le 2$. There is no restriction on θ so $0 \le \theta \le 2\pi$.

$$\iiint_E x^2 + y^2\,dV = \int_0^{2\pi} \int_0^2 \int_{-1}^2 r^2 \cdot r\,dz\,dr\,d\theta = \int_0^{2\pi} d\theta \cdot \int_0^2 r^3\,dr \cdot \int_{-1}^2 dz$$

$$= 2\pi \cdot \frac{1}{4}2^4 \cdot 3 = \boxed{24\pi.}$$

6. (12 points) Consider the region $R = \{(x,y) \mid 1 \le x^2+y^2 \le 4 \text{ and } x \ge 0\}$. Sketch this region then find its centroid. Recall that...

$$(\bar{x},\bar{y}) = \frac{1}{m}(M_y, M_x) \qquad m = \iint_R 1\,dA \qquad M_y = \iint_R x\,dA \qquad M_x = \iint_r y\,dA$$



Notice that this region is nothing more than the right-half (i.e. $x \ge 0$) of an annulus (a region between 2 circles). Also, notice that we get $\bar{y} = 0$ by symmetry.

Next, $m = \dfrac{\pi \cdot 2^2 - \pi \cdot 1^2}{2} = \dfrac{3\pi}{2}$ since the area of an annulus is the difference between the areas of the circles defining it (and don't forget we're only dealing with half).

To find $\bar{x}$, we'll need to compute the moment about the y-axis. The double integral defining M_y is best dealt with in terms of polar coordinates where our annular region is described by $1 \le r \le 2$ and $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.

$$M_y = \iint_R x\,dA = \int_{-\pi/2}^{\pi/2}\int_1^2 r\cos(\theta)\cdot r\,dr\,d\theta = \int_{-\pi/2}^{\pi/2}\cos(\theta)\,d\theta \cdot \int_1^2 r^2\,dr = 2\cdot\left[\frac{r^3}{3}\bigg|_1^2\right] = \frac{2}{3}\left(2^3-1^3\right) = \frac{14}{3}$$

Notice that we could "factor" our integral since $r^2\cos(\theta)$ factors into r and θ parts and we have only constant bounds.

Finally, $\bar{x} = \dfrac{M_y}{m} = \dfrac{14/3}{3\pi/2} = \dfrac{28}{9\pi}$. Therefore, $\boxed{(\bar{x},\bar{y}) = \left(\dfrac{28}{9\pi}, 0\right)}$

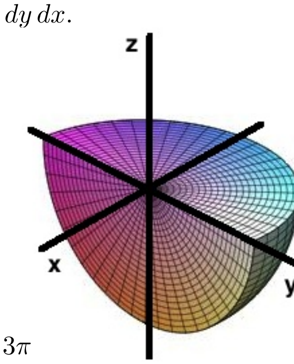
7. (12 points) Compute $\displaystyle\iint_R x\,dA$ where R is the region bounded by $y=-x$, $y=-x+1$, $y=2x$, and $y=2x+2$. Use a (natural) change of coordinates which simplifies the region R and... don't forget the Jacobian!

Notice we can restate our bounds as follows: $x+y=0$, $x+y=1$, $-2x+y=0$ and $-2x+y=2$. These bounds suggest a natural change of coordinates, that is $u = x+y$ and $v = -2x+y$. With this substitution, the u,v bounds are easy to see from our restatement of the x,y bounds, that is $0 \le u \le 1$ and $0 \le v \le 2$. Before we restate and compute the integral, we should not forget the Jacobian! (We we're even warned this time!) Now $J = \dfrac{\partial(x,y)}{\partial(u,v)}$. In order to find this, we need functions x and y in terms of u and v as opposed to what we have, which is functions u and v in terms of x and y. Now, we may certainly solve our u and v functions for x and y in order to compute the Jacobian, for example, $u - v = (x+y) - (-2x+y) = 3x \implies x = \dfrac{1}{3}u - \dfrac{1}{3}v$ and $2u + v = 3y \implies y = \frac{2}{3}u + \frac{1}{3}v$. This implies that $J = \dfrac{\partial(x,y)}{\partial(u,v)} = \det\begin{bmatrix} x_u & y_u \\ x_v & y_v \end{bmatrix} = \det\begin{bmatrix} 1/3 & 2/3 \\ -1/3 & 1/3 \end{bmatrix} = \frac{1}{9} - \frac{-2}{9} = \frac{1}{3}$.

However, we have a slicker way yet. Recall that $\dfrac{1}{\frac{\partial(u,v)}{\partial(x,y)}} = \frac{\partial(x,y)}{\partial(u,v)}$. As our u,v variables are already in terms of x and y we can simply compute J^{-1} and take the reciprocal. $\dfrac{\partial(u,v)}{\partial(x,y)} = \det\begin{bmatrix} u_x & v_x \\ u_y & v_y \end{bmatrix} = \det\begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} = 3 = J^{-1}$ Hence, $J = \dfrac{1}{J^{-1}} = \dfrac{1}{3}$. Armed with our Jacobian, we are ready to substitute variables and evaluate our integral. Note, we did in fact need to solve for x but not y.

$$\iint_R x\,dA = \int_0^2\int_0^1 \left(\frac{u-v}{3}\right)\frac{1}{3}\,du\,dv = \frac{1}{9}\int_0^2\int_0^1 u - v\,du\,dv = \frac{1}{9}\int_0^2 \frac{1}{2} - v\,dv = \frac{1}{9}\left[1 - \frac{1}{2}(2)^2\right] = \frac{1}{9}[-1] = \boxed{\frac{-1}{9}}.$$

8. (12 points) Consider the integral: $I = \displaystyle\int_{-3}^{0}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_{-\sqrt{9-x^2-y^2}}^{0} z\cos(x^2+y^2+z^2)\,dz\,dy\,dx.$



This region is bounded by $-\sqrt{9-x^2-y^2} \le z \le 0$, $-\sqrt{9-x^2} \le y \le \sqrt{9-x^2}$, and $-3 \le x \le 0$. The first pair of inequalities say that z is bounded between the lower-half of a sphere of radius 3 and the xy-plane. The next inequalities say that y is bounded by the circle of radius 3, centered at the origin lying in xy-plane. Finally, x ranges from -3 to 0. Putting this together we get that our region of integration is the back-half of the lower-half of the solid ball of radius 3 (centered at the origin).

When switching to cylindrical coordinates, don't forget that $x^2+y^2 = r^2$ and the Jacobian is r. Also, obviously since (x,y) are trapped inside the circle of radius 3 (i.e. $x^2+y^2 \le 9$) we get $0 \le r \le 3$ and the back-half of the circle corresponds to $\dfrac{\pi}{2} \le \theta \le \dfrac{3\pi}{2}$.

In spherical coordinates, we have that $x^2+y^2+z^2 = \rho^2$ and the Jacobian is $\rho^2\sin(\varphi)$. Also, $x^2+y^2+z^2 \le 9$ (inside the sphere) translates to $0 \le \rho \le 3$ and the lower-half of 3-space corresponds to $\pi/2 \le \varphi \le \pi$.

(a) Rewrite I in the following order of integration: $\iiint \quad dx\,dz\,dy$.

Do **not** evaluate the integral.

$$\int_{-3}^{3}\int_{-\sqrt{9-y^2}}^{0}\int_{-\sqrt{9-y^2-z^2}}^{0} z\cos(x^2+y^2+z^2)\,dx\,dz\,dy.$$

(b) Rewrite I in terms of cylindrical coordinates.

Do **not** evaluate the integral.

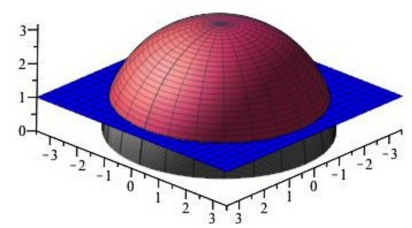
$$\int_{\pi/2}^{3\pi/2}\int_{0}^{3}\int_{-\sqrt{9-r^2}}^{0} z\cos(r^2+z^2)\cdot r\,dz\,dr\,d\theta.$$

(c) Rewrite I in terms of spherical coordinates.

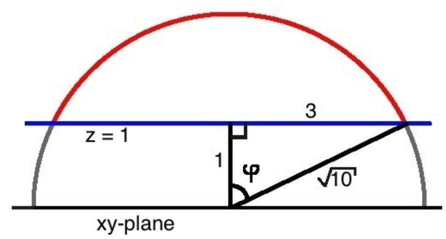
Do **not** evaluate the integral.

$$\int_{\pi/2}^{3\pi/2}\int_{\pi/2}^{\pi}\int_{0}^{3} \rho\cos(\varphi)\cdot\cos(\rho^2)\cdot\rho^2\sin(\varphi)\,d\rho\,d\varphi\,d\theta.$$

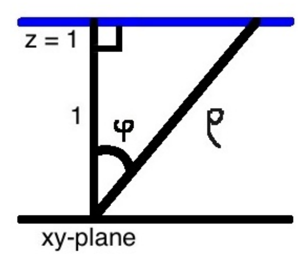
9. (12 points) Let E be the region below the hemisphere $z=\sqrt{10-x^2-y^2}$ and above the plane $z=1$.



This region is the top portion of a sphere of radius $\sqrt{10}$ cut-off below by the plane $z=1$ (as pictured on the right). When considering cylindrical and spherical coordinates it is easy to see that $0\le\theta\le 2\pi$. The bounds for z are easy as well since they are given those in the problem: $z=1$ and $z=\sqrt{10-x^2-y^2}=\sqrt{10-r^2}$. Thus $1\le z\le\sqrt{10-r^2}$. We are left to find the bounds for r. To do this, we simply need to consider where the hemisphere $z=\sqrt{10-r^2}$ and the plane $z=1$ intersect. Setting these equations equal we see $1=\sqrt{10-r^2} \implies 1=10-r^2 \implies r^2=9 \implies r=3$. Hence, $0\le r\le 3$.



Now, with respect to spherical coordinates. Again, we have that $0\le\theta\le 2\pi$. For φ, we consider the triangle formed between the radius of the hemisphere, $z=1$ (where above we found $r=3$) and the z-axis (see the figure to the right). φ at this point will give us an upper bound. There are many ways to describe this angle: $\varphi=\arctan(3)=\arcsin\left(\frac{3}{\sqrt{10}}\right)=\arccos\left(\frac{1}{\sqrt{10}}\right)$. Thus we have $0\le\varphi\le\arctan(3)$. This leaves us to find the bounds for ρ. Notice that ρ ranges from the plane $z=1$ to the hemisphere $z=\sqrt{10-x^2-y^2}$. The hemisphere is simply $\rho=\sqrt{10}$ (the radius of our sphere). The ρ's lower bound is determined by $z=1$ which is $1=z=\rho\cos(\varphi)$ so $\rho=\sec(\varphi)$. Thus, we have $\sec(\varphi)\le\rho\le\sqrt{10}$.



We integrate over 1 to compute the volume and, finally, *don't forget the Jacobians!*

(a) Write down an integral which computes the volume of E in cylindrical coordinates. Do not evaluate this integral.

$$\int_{0}^{2\pi}\int_{0}^{3}\int_{1}^{\sqrt{10-r^2}} 1\cdot r\,dz\,dr\,d\theta.$$

(b) Write down an integral which computes the volume of E in spherical coordinates. Do not evaluate this integral.

$$\int_{0}^{2\pi}\int_{0}^{\arctan(3)}\int_{\sec(\varphi)}^{\sqrt{10}} 1\cdot\rho^2\sin(\varphi)\,d\rho\,d\varphi\,d\theta.$$