Math 2130-101 # Test #1 **July 8th, 2015**

Name: Answer Key **Be sure to show your work!**

1. (17 points) Vector Basics: Let $\mathbf{u} = \langle 2, 1, -1\rangle$, $\mathbf{v} = \langle 3, 1, 2\rangle$, and $\mathbf{w} = \langle -1, 2, -2\rangle$.

(a) Compute the area of the parallelogram spanned by $\mathbf{v}$ and $\mathbf{w}$.

$$\mathbf{v}\times\mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 1 & 2 \\ -1 & 2 & -2 \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ 2 & -2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 3 & 2 \\ -1 & -2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 3 & 1 \\ -1 & 2 \end{vmatrix}\mathbf{k} = (1(-2)-2(2))\mathbf{i} - (3(-2)-(-1)(2))\mathbf{j} + (3(2)-(-1)1)\mathbf{k}$$

$$\mathbf{v}\times\mathbf{w} = \langle -6, 4, 7\rangle \quad\Longrightarrow\quad |\mathbf{v}\times\mathbf{w}| = \sqrt{36+16+49} = \boxed{\sqrt{101}}$$

(b) Find the volume of the parallelepiped spanned by $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$.

$$\mathbf{u}\bullet(\mathbf{v}\times\mathbf{w}) = \langle 2,1,-1\rangle\bullet\langle -6,4,7\rangle = 2(-6)+1(4)+(-1)7 = -15 \quad\Longrightarrow\quad \text{The volume is } |-15| = \boxed{15}.$$

(c) Find two vectors of length 5 which are parallel to $\mathbf{w}$.

First, normalize $\mathbf{w}$: $\dfrac{\mathbf{w}}{|\mathbf{w}|} = \dfrac{1}{\sqrt{(-1)^2+2^2+(-2)^2}}\langle -1,2,-2\rangle = \dfrac{1}{3}\langle -1,2,-2\rangle$. So now we have a *unit* vector which points in the same direction as $\mathbf{w}$. If we want a vector of length 5, we just need to scale this vector by 5 (chucking in a minus sign will yield the second desired vector): $\boxed{\pm\frac{5}{3}\langle -1,2,-2\rangle}$

(d) Find the angle between $\mathbf{v}$ and $\mathbf{w}$ (don't worry about evaluating inverse trig. functions).

First, $\mathbf{v}\bullet\mathbf{w} = 3(-1)+1(2)+2(-2) = -5$, $|\mathbf{v}| = \sqrt{3^2+1^2+2^2} = \sqrt{14}$, and $|\mathbf{w}| = \sqrt{(-1)^2+2^2+(-2)^2} = 3$. We know that $\mathbf{v}\bullet\mathbf{w} = |\mathbf{v}||\mathbf{w}|\cos(\theta)$. Thus $-5 = 3\sqrt{14}\cos(\theta)$. Therefore, $\boxed{\theta = \arccos\left(\frac{-5}{3\sqrt{14}}\right)}$. Since $\mathbf{v}\bullet\mathbf{w} = -5 < 0$, the angle is obtuse.

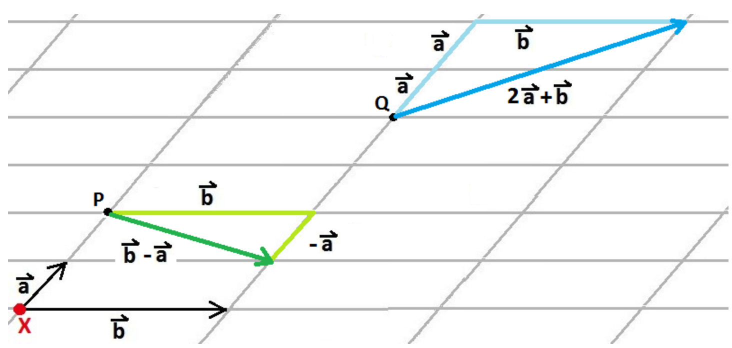
Is this angle... **right,** **acute,** or **obtuse** ? (Circle your answer.)

(e) Match the statement on the left to the corresponding statement on the right...

A $(\mathbf{a}\times\mathbf{b})\bullet\mathbf{b} = 0$	**D)** $\mathbf{a}$ and $\mathbf{b}$ are parallel
C $(\mathbf{a}\bullet\mathbf{b})\times(\mathbf{a}\bullet\mathbf{b}) = \mathbf{0}$	**B)** $\mathbf{a}$ and $\mathbf{b}$ are orthogonal
D $\mathbf{a}\times\mathbf{b} = \mathbf{0}$	**A)** is always true
B $\mathbf{a}\bullet\mathbf{b} = 0$	**C)** is nonsense

Note: $(\mathbf{a}\bullet\mathbf{b})\times(\mathbf{a}\bullet\mathbf{b})$ is nonsense since dot products yield scalars and you can't cross product scalars.

(f) The vectors $\mathbf{a}$ and $\mathbf{b}$ are shown to the right. They are based at the point X. Sketch the vector $\mathbf{b}-\mathbf{a}$ based at the point P and sketch the vector $2\mathbf{a}+\mathbf{b}$ based at the point Q.



2. (10 points) Let ℓ_1 be parametrized by $\mathbf{r}_1(t) = \langle 3,-1,-1\rangle + \langle 1,-1,1\rangle t$ and let ℓ_2 be the line which passes through the points $P = (2,1,-1)$ and $Q = (3,1,1)$. Determine if ℓ_1 and ℓ_2 are... (circle the correct answer)

the same, parallel (but not the same), **intersecting**, or skew.

The line through P and Q has direction vector $\vec{PQ} = Q - P = \langle 3-2, 1-1, 1-(-1)\rangle = \langle 1,0,2\rangle$. Thus ℓ_2 is parameterized by $\mathbf{r}_2(t) = \langle 2,1,-1\rangle + t\langle 1,0,2\rangle$.

The direction vector for $\mathbf{r}_1(t)$ is $\mathbf{r}_1' = \langle 1,-1,1\rangle$. This vector is not a multiple of $\mathbf{r}_2' = \langle 1,0,2\rangle$. Thus these lines are not the same or parallel. Let's see if they intersect (don't forget to use 2 different parameters): $\mathbf{r}_1(t) = \mathbf{r}_2(s)$. Thus $\mathbf{r}_1(t) = \langle 3+t, -1-t, -1+t\rangle = \mathbf{r}_2(s) = \langle 2+s, 1, -1+2s\rangle$. Stripping apart into components we get that $3+t = 2+s$, $-1-t = 1$, and $-1+t = -1+2s$. The second equation ($-1-t=1$) yields $t=-2$. Plugging this into the first equation, we get $3+(-2) = 2+s$ so that $s=-1$. Notice that $\mathbf{r}_1(-2) = \langle 1,1,-3\rangle = \mathbf{r}_2(-1)$. Therefore, these lines intersect at the point $(1,1,-3)$.

3. (14 points) Plane old geometry.

(a) Find the (scalar) equation of the plane through the points $A = (2, 1, 0)$, $B = (3, 3, 1)$, and $C = (2, 3, -1)$.

The vectors $\vec{AB} = B - A = \langle 3-2, 3-1, 1-0\rangle = 1, 2, 1\rangle$ and $\vec{AC} = C - A = \langle 2-2, 3-1, -1-0\rangle = \langle 0, 2, -1\rangle$ are parallel to the plane (because A, B, C lie in the plane). Thus $\mathbf{n} = \vec{AB} \times \vec{AC}$ will be orthogonal to the plane.

$$\mathbf{n} = \vec{AB} \times \vec{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 1 \\ 0 & 2 & -1 \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 2 & -1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 1 \\ 0 & -1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 2 \\ 0 & 2 \end{vmatrix}\mathbf{k} = \langle -4, 1, 2\rangle$$

Using the first point $A = (2, 1, 0$, we get $-4(x-2)+1(y-1)+2(z-0) = 0$. Multiplying out yields $\boxed{-4x + y + 2z + 7 = 0}$.

(b) Find the area of the triangle with vertices A, B, and C (from part (a)).

The parallelogram with A, B, C, and some fourth point D is spanned by $\vec{AB}$ and $\vec{AC}$. It has area $|\vec{AB} \times \vec{AC}| = |\langle -4, 1, 2\rangle| = \sqrt{16+1+4} = \sqrt{21}$. The triangle ΔABC has area half that of this parallelogram. Therefore, the area of the triangle is $\boxed{\sqrt{21}/2}$.

(c) The planes: $3x - 2y + 2z = 10$ and $2x - y - 4z = 7$ are... [Circle **ALL** that apply.]

parallel   [perpendicular]   [intersecting]   the same

Normal vectors for these planes are $\mathbf{n}_1 = \langle 3, -2, 2\rangle$ and $\mathbf{n}_2 = \langle 2, -1, -4\rangle$ respectively. Notice that $\mathbf{n}_1 \bullet \mathbf{n}_2 = 3(2) + (-2)(-1) + 2(-4) = 0$. Thus these normal vectors are perpendicular, so the planes are as well. Obviously this means that the planes must intersect (and aren't parallel or the same).

4. (8 points) Is the curvature of $y = e^{-2x}$ ever zero?   Yes / [No]

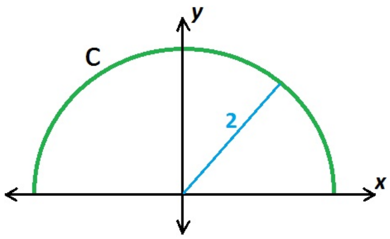
Here we have the graph of a function, so we can use our special formula for curvature. First, $y' = -2e^{-2x}$ and so $y'' = 4e^{-2x}$.

$$\kappa(x) = \frac{|y''|}{(1+(y')^2)^{3/2}} = \frac{4e^{-2x}}{(1+4e^{-4x})^{3/2}}$$

Notice that $\kappa(x) \neq 0$ since e^{-2x} is never zero (the exponential function is always positive).

5. (12 points) Parameterization, arc length, and a line integral.

(a) Let C be the upper-half of the circle $x^2 + y^2 = 4$. Parameterize C and then compute its centroid. [*Hint:* Take advantage of geometry and symmetry.]



First, we (typically) parameterize circles with sine and cosine: $\mathbf{r}(t) = \langle 2\cos(t), 2\sin(t)\rangle$ (use 2 since the radius of the circle is 2). We just want the upper-half, so recalling that t is really an angle swept out from the x-axis, we should have $0 \le t \le \pi$ to just get the upper-half of the circle.
Next, $m = \int_C 1\, ds$ is just arc length. We have half of a circle so arc length is just π times the radius: $m = 2\pi$. Also, by symmetry $\bar{x} = 0$. This means we only have one line intergral to compute: $M_x = \int_C y\, ds$.

To compute a line integral with respect to arc length we need a parameterization for our curve (done) and then need to compute the arc length element: ds. $\mathbf{r}'(t) = \langle -2\sin(t), 2\cos(t)\rangle$ so that $|\mathbf{r}'(t)| = \sqrt{4\sin^2(t) + 4\cos^2(t)} = 2$. Thus $ds = 2\, dt$.

$M_x = \displaystyle\int_C y\, ds = \int_0^\pi 2\sin(t)\cdot 2\, dt = 4\int_0^\pi \sin(t)\, dt = 4 \cdot 2 = 8$. Therefore, $\bar{y} = \dfrac{M_x}{m} = \dfrac{8}{2\pi} = \dfrac{4}{\pi}$ (which is about one and a third).

Therefore, the centroid of C is $\boxed{(\bar{x}, \bar{y}) = \left(0, \frac{4}{\pi}\right)}$

6. (15 points) Let C be parameterized by $\mathbf{r}(t) = \langle \sin(t), t^3, e^t\rangle$ where $-3 \le t \le 1$.

(a) Set up the line integral $\int_C (x^2 e^y + 4z)\, ds$.   [Do not try to evaluate this integral. It will only end in tears.]

First, we need to compute ds. $\mathbf{r}'(t) = \langle \cos(t), 3t^2, e^t\rangle$. Thus $|\mathbf{r}'(t)| = \sqrt{\cos^2(t) + 9t^4 + e^{2t}}$.

$$\int_C (x^2 e^y + 4z)\, ds = \boxed{\int_{-3}^{1} (\sin^2(t) e^{t^3} + 4e^t)\sqrt{\cos^2(t) + 9t^4 + e^{2t}}\, dt}$$

(b) Find the curvature of $\mathbf{r}(t)$.

We have $\mathbf{r}'(t) = \langle \cos(t), 3t^2, e^t \rangle$ and so $\mathbf{r}''(t) = \langle -\sin(t), 6t, e^t \rangle$.

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos(t) & 3t^2 & e^t \\ -\sin(t) & 6t & e^t \end{vmatrix} = \begin{vmatrix} 3t^2 & e^t \\ 6t & e^t \end{vmatrix} \mathbf{i} - \begin{vmatrix} \cos(t) & e^t \\ -\sin(t) & e^t \end{vmatrix} \mathbf{j} + \begin{vmatrix} \cos(t) & 3t^2 \\ -\sin(t) & 6t \end{vmatrix} \mathbf{k}$$

$\mathbf{r}' \times \mathbf{r}'' = \langle (3t^2-6)e^t, -e^t(\cos(t)+\sin(t)), 6t\cos(t)+3t^2\sin(t) \rangle$ and so

$$\kappa(t) = \frac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|^3} = \boxed{\frac{\sqrt{(3t^2-6)^2e^{2t} + e^{2t}(\cos(t)+\sin(t))^2 + (6t\cos(t)+3t^2\sin(t))^2}}{(\cos^2(t)+9t^4+e^{2t})^{3/2}}}$$

(c) Find the tangential and normal components of acceleration for $\mathbf{r}(t)$.

$$a_T(t) = \frac{\mathbf{r}' \bullet \mathbf{r}''}{|\mathbf{r}'|} = \boxed{\frac{-\cos(t)\sin(t) + 18t^3 + e^{2t}}{\sqrt{\cos^2(t)+9t^4+e^{2t}}}} \qquad \text{(tangential component)}$$

$$a_N(t) = \frac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|} = \boxed{\frac{\sqrt{(3t^2-6)^2e^{2t} + e^{2t}(\cos(t)+\sin(t))^2 + (6t\cos(t)+3t^2\sin(t))^2}}{\sqrt{\cos^2(t)+9t^4+e^{2t}}}} \qquad \text{(normal component)}$$

7. (12 points) Find the TNB-frame for $\mathbf{r}(t) = \langle 4\sin(t), 4\cos(t), 3t \rangle$.

$\mathbf{r}'(t) = \langle 4\cos(t), -4\sin(t), 3 \rangle$ so $|\mathbf{r}'(t)| = \sqrt{16\cos^2(t) + 16\sin^2(t) + 9} = \sqrt{16+9} = 5$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \boxed{\frac{1}{5}\left\langle 4\cos(t), -4\sin(t), 3 \right\rangle}$$

$\mathbf{T}'(t) = \frac{1}{5}\langle -4\sin(t), -4\cos(t), 0 \rangle = \frac{4}{5}\langle -\sin(t), -\cos(t), 0 \rangle$ so $|\mathbf{T}'(t)| = \frac{4}{5}\sqrt{\sin^2(t)+\cos^2(t)} = \frac{4}{5}$

(We don't need this, but at this point we could note that $\kappa = |\mathbf{T}'|/|\mathbf{r}'| = (4/5)/(1/5) = \frac{4}{25}$.)

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \frac{\frac{4}{5}\langle -\sin(t), -\cos(t) \rangle}{\frac{4}{5}} = \boxed{\langle -\sin(t), -\cos(t), 0 \rangle}$$

$$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ (4/5)\cos(t) & -(4/5)\sin(t) & (3/5) \\ -\sin(t) & -\cos(t) & 0 \end{vmatrix}$$

$$= \begin{vmatrix} -(4/5)\sin(t) & (3/5) \\ -\cos(t) & 0 \end{vmatrix} \mathbf{i} - \begin{vmatrix} (4/5)\cos(t) & (3/5) \\ -\sin(t) & 0 \end{vmatrix} \mathbf{j} + \begin{vmatrix} (4/5)\cos(t) & -(4/5)\sin(t) \\ -\sin(t) & -\cos(t) \end{vmatrix} \mathbf{k}$$

$$= \left\langle \frac{3}{5}\cos(t), -\frac{3}{5}\sin(t), -\frac{4}{5}\cos^2(t) - \frac{4}{5}\sin^2(t) \right\rangle = \boxed{\frac{1}{5}\left\langle 3\cos(t), -3\sin(t), -4 \right\rangle}$$

Does this curve lie in a plane? Why or why not?

No. Notice that the binormal $\mathbf{B}(t)$ is not constant. This means that our curve is not a planar curve. In fact, $\mathbf{r}(t)$ parameterizes a helix.

8. (12 points) No numbers here. Choose **ONE** of the following:

I. Suppose that $|\mathbf{r}(t)| = c$ (the length of $\mathbf{r}(t)$ is constant). Prove that $\mathbf{r}(t)$ and $\mathbf{r}'(t)$ are orthogonal.

We get $c^2 = |\mathbf{r}(t)|^2 = \mathbf{r}(t) \bullet \mathbf{r}(t)$. c^2 is constant so its derivative (with respect to t) is 0. Thus (using the product rule) we get $0 = \mathbf{r}'(t) \bullet \mathbf{r}(t) + \mathbf{r}(t) \bullet \mathbf{r}'(t)$ so $0 = 2\,\mathbf{r}(t) \bullet \mathbf{r}'(t)$. Thus $\mathbf{r}(t) \bullet \mathbf{r}'(t) = 0$ (they are orthogonal).

II. Suppose that $\mathbf{a}$ and $\mathbf{b}$ are unit vectors. Prove that $(\mathbf{a} \bullet \mathbf{b})^2 + |\mathbf{a} \times \mathbf{b}|^2 = 1$. [*Suggestion:* Use a fundamental identity for both $\mathbf{a} \bullet \mathbf{b}$ and $|\mathbf{a} \times \mathbf{b}|$. Don't try to prove this with components.]

We use the following facts: $|\mathbf{a}| = |\mathbf{b}| = 1$ (they are unit vectors), $\mathbf{a} \bullet \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos(\theta)$, and $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin(\theta)$ where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$. Therefore, $(\mathbf{a} \bullet \mathbf{b})^2 + |\mathbf{a} \times \mathbf{b}|^2 = |\mathbf{a}|^2|\mathbf{b}|^2\cos^2(\theta) + |\mathbf{a}|^2|\mathbf{b}|^2\sin^2(\theta) = \cos^2(\theta) + \sin^2(\theta) = 1$.