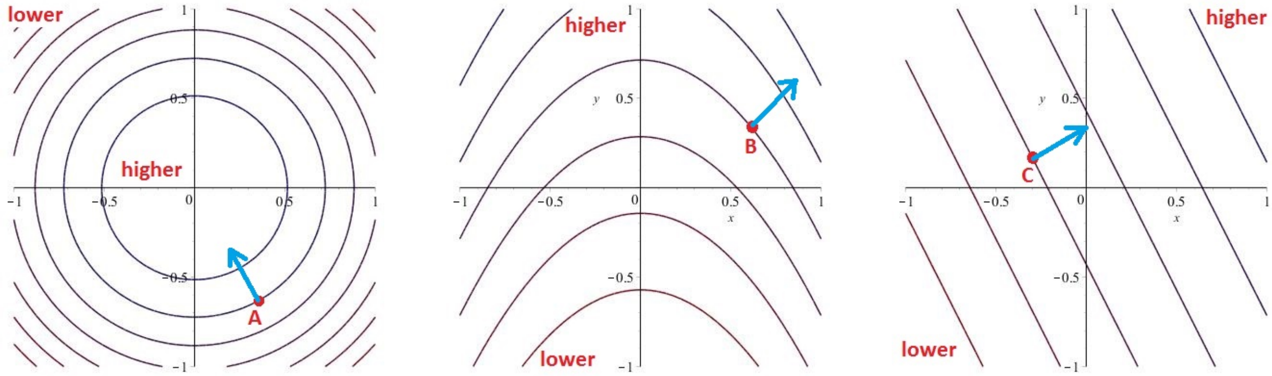
Test #2

Name: ANSWER KEY

Be sure to show your work!

1. (11 points) Three level curve plots are shown below. I have labeled the levels so you know which curves are higher and which are lower.



$f(x,y) = \sqrt{9 - x^2 - y^2}$ | $f(x,y) = x^2 + y$ | $f(x,y) = 2x + y$

(a) The plots above correspond to 3 of the functions listed here: $f(x,y) = \sqrt{9-x^2-y^2}$, $f(x,y) = 3\sqrt{x^2+y^2}$, $f(x,y) = 2x+y$, $f(x,y) = x^2+y$, and $f(x,y) = -x^2+y$. Write the correct formula below each plot.

The first plot's level curves are clearly circles. Of the formulas listed, the first two have circular level curves. For the first formula: $\sqrt{9-x^2-y^2} = C$ we have $x^2+y^2 = 9 - C^2$ so that the circles get smaller as C gets larger (this matches our graph). On the other hand, for the second formula: $3\sqrt{x^2+y^2} = C$ we have $x^2+y^2 = C/9$ so the circles grow as C gets larger (this doesn't match). Thus our first plot goes with $f(x,y) = \sqrt{9-x^2-y^2}$.

The second plot's level curves look like parabolas. The last two formulas have parabolic level curves. For $f(x,y) = x^2+y$ we get $x^2+y = C$ so that $y = -x^2+C$ (parabolas opening downward – matching our graph). On the other hand, for $f(x,y) = -x^2+y$, we get $-x^2+y = C$ so that $y = x^2+C$ (parabolas opening upward – this doesn't match). Thus our second plot goes with $f(x,y) = x^2+y$.

Finally, the third plot's level curves are lines. The only formula with such level curves is $f(x,y) = 2x+y$ (we have $2x+y = C$ so $y = -2x+C$). This must be the correct formula.

(b) Sketch a gradient vector at the points A, B, and C. If the vector is $\mathbf{0}$, draw an "X" on the point.
[Don't worry about having the correct length. I'm just looking for the correct direction.]

Remember that gradients are orthogonal to level curves and point toward "higher ground".

(c) If A, B, or C is a critical point, indicate what kind of point it is (i.e. local min, local max, saddle, or other).

To be a critical point we need $\nabla f = \mathbf{0}$ or ∇f to be undefined. We have non-zero gradients at each point, so none of these are critical points.

2. (8 points) Let $z = f(x,y)$ where $x = u+v$ and $y = u-v$. Show that $\dfrac{\partial z}{\partial u} + \dfrac{\partial z}{\partial v} = 2\dfrac{\partial z}{\partial x}$.

Notice that

$$\frac{\partial x}{\partial u} = \frac{\partial}{\partial u}\Big[u+v\Big] = 1, \qquad \frac{\partial x}{\partial v} = \frac{\partial}{\partial v}\Big[u+v\Big] = 1, \qquad \frac{\partial y}{\partial u} = \frac{\partial}{\partial u}\Big[u-v\Big] = 1, \qquad \text{and} \qquad \frac{\partial y}{\partial v} = \frac{\partial}{\partial v}\Big[u-v\Big] = -1$$

Now just write down the chain rule...

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} = \frac{\partial z}{\partial x}(1) + \frac{\partial z}{\partial y}(1) \qquad \text{and} \qquad \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} = \frac{\partial z}{\partial x}(1) + \frac{\partial z}{\partial y}(-1)$$

Put this together and get...

$$\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} = \left(\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\right) + \left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right) = 2\frac{\partial z}{\partial x}$$

3. (10 points) Let $\ln(xy^2z^3+1)+x^3y+z^3=8$.

(a) Find an equation for the plane tangent to the above surface at the point $(x,y,z)=(-1,0,2)$.

Letting $F(x,y,z)=\ln(xy^2z^3+1)+x^3y+z^3$. We are considering the level surface $F(x,y,z)=8$. We know that the gradient ∇F gives us a formula for normals to tangent planes of level surfaces of $F(x,y,z)$. This we need to compute $\nabla F(-1,0,2)$.

$$\nabla F = \langle F_x, F_y, F_z\rangle = \left\langle \frac{y^2z^3}{xy^2z^3+1}+3x^2y,\ \frac{2xyz^3}{xy^2z^3+1}+x^3,\ \frac{3xy^2z^2}{xy^2z^3+1}+3z^2 \right\rangle$$

Therefore, $\nabla F(-1,0,2) = \left\langle \frac{0}{1}+0,\ \frac{0}{1}+(-1)^3,\ \frac{0}{1}+3(2^2) \right\rangle = \langle 0,-1,12\rangle$.

Thus the tangent plane is: $\boxed{0(x+1)-1(y-0)+12(z-2)=0}$ or $\boxed{-y+12z-24=0}$.

(b) Considering z as a variable depending on x and y (defined implicitly above), find $\dfrac{\partial z}{\partial x}$.

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = \boxed{-\frac{\dfrac{y^2z^3}{xy^2z^3+1}+3x^2y}{\dfrac{3xy^2z^2}{xy^2z^3+1}+3z^2}} = \frac{-3x^3y^3z^3-3x^2y-y^2z^3}{3xy^2z^5+3xy^2z^2+3z^2}$$

[I simiplified the answer, but you didn't need to.]

4. (10 points) Limits

(a) Show the following limit **does** exist: $\displaystyle\lim_{(x,y)\to(0,0)} \frac{5x^3+4x^2+4y^2+3y^5}{x^2+y^2}$

We should switch to polar coordinates: $x=r\cos(\theta)$, $y=r\sin(\theta)$, and so $x^2+y^2=r^2$. Then we get...

$$\lim_{(x,y)\to(0,0)} \frac{5x^3+4x^2+4y^2+3y^5}{x^2+y^2} = \lim_{(r,\theta)\to(0,\theta)} \frac{5r^3\cos^3(\theta)+4r^2+3r^5\sin^5(\theta)}{r^2}$$

$$= \lim_{(r,\theta)\to(0,\theta)} \frac{r^2(5r\cos^3(\theta)+4+3r^3\sin^5(\theta))}{r^2} = \lim_{(r,\theta)\to(0,\theta)} 5r\cos^3(\theta)+4+3r^3\sin^5(\theta) = 0+4+0 = \boxed{4}$$

[*Note:* Since cosine and sine are bounded between -1 and 1, we have (for example) $-5r \le 5r\cos^3(\theta) \le 5r$ so (by the squeeze theorem) $5r\cos^3(\theta)\to 0$ since $\pm 5r \to 0$ as $r\to 0$.]

(b) Show the following limit **does not** exist: $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2x^2y}{x^4+y^2}$. *Hint:* Unify the denominator.

If we approach along $y=0$ (the x-axis which passes through the origin), we get $\displaystyle\lim_{x\to 0}\frac{2x^2(0)}{x^4+0^2} = \lim_{x\to 0}\frac{0}{x^4} = 0$.

We could try approaching along $x=0$, $y=x$, etc. but any line through the origin will also yield an answer of 0 (not helpful), so we should try the hint.

If we approach along $y=x^2$ (this makes $x^4+y^2=x^4+(x^2)^2=2x^4$), we get $\displaystyle\lim_{x\to 0}\frac{2x^2(x^2)}{x^4+(x^2)^2} = \lim_{x\to 0}\frac{2x^4}{2x^4} = 1$.

Thus approaching along $y=0$ and $y=x^2$ give different answers, so this limit cannot exist.

5. (9 points) Suppose we have a function of two variables: $f(x,y)$.

(a) It is possible for $f_{xy}(3,5)=-1$ and $f_{yx}(3,5)=2$? If not, why not. If so, what does this tell us?

$\boxed{\textbf{Yes}}$. Clairaut's theorem guarantees that mixed partials are equal if they are **continuous**. If f_{xy} and f_{yx} are discontinuous, we can have $f_{xy}\neq f_{yx}$.

(b) Suppose f_x and f_y are continuous everywhere. Can I conclude f is continuous? $\boxed{\textbf{YES}}$

We have a theorem which says "continuous partials implies differentiability" and another theorem which says "differentiability implies continuity of the function". Putting these two theorems together we get continuous partials implies that we have a continuous function.

(c) Suppose f_x and f_y exist everywhere. Can I conclude that f is differentiable? $\boxed{\textbf{NO}}$

While differentiability implies the existance of first partials, the converse is not true. Essentially differentiability amounts to the existance of a multivariate limit while existance of partials amounts to that limit existing along coordinate axes. Existence of partials is strictly weaker than differentiability.

6. (12 points) Let $f(x,y) = x^3 - 3x - y^3 + 12y$.

(a) Find the gradient of f and the Hessian matrix of f.

$$\nabla f = \langle f_x, f_y \rangle = \boxed{\left\langle 3x^2 - 3, -3y^2 + 12 \right\rangle} \qquad \text{and} \qquad H_f = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \boxed{\begin{bmatrix} 6x & 0 \\ 0 & -6y \end{bmatrix}}$$

(b) Find the quadratic approximation of f at $(x,y) = (2,-1)$.

We need to plug in our point: $f(2,-1) = 2^3 - 3(2) - (-1)^3 + 12(-1) = -9$, $\nabla f(2,-1) = \langle 3(2^2) - 3, -3(-1)^2 + 12 \rangle = \langle 9, 9 \rangle$, and $H_f(2,-1) = \begin{bmatrix} 12 & 0 \\ 0 & 6 \end{bmatrix}$.

$$Q(x,y) = -9 + \langle 9,9 \rangle \bullet \langle x-2, y+1 \rangle + \frac{1}{2} \begin{bmatrix} x-2 & y+1 \end{bmatrix} \begin{bmatrix} 12 & 0 \\ 0 & 6 \end{bmatrix} \begin{bmatrix} x-2 \\ y+1 \end{bmatrix}$$

...OR...

$$Q(x,y) = -9 + 9(x-2) + 9(y+1) + \frac{12}{2}(x-2)^2 + \frac{0}{2}(x-2)(y+1) + \frac{0}{2}(x-2)(y+1) + \frac{6}{2}(y+1)^2$$

...OR...

$$Q(x,y) = -9 + 9(x-2) + 9(y+1) + 6(x-2)^2 + 3(y+1)^2$$

(c) Find an classify the critical point(s) of $f(x,y)$.
[Use the "2^{nd}-derivative" test to determine if critical points are relative max's, min's or saddle points.]

First, $\nabla f(x,y)$ is defined everywhere, so we don't get critical points from the derivative failing to exist. Next, we need to know where $\nabla f(x,y) = \langle 0,0 \rangle$. Thus we need to solve the system of equations: $3x^2 - 3 = 0$ and $-3y^2 + 12 = 0$. So $x^2 = 1$ and $y^2 = 4$. Therefore, $x = \pm 1$ and $y = \pm 2$. There are 4 critical points.

- $H_f(-1,-2) = \begin{bmatrix} -6 & 0 \\ 0 & 12 \end{bmatrix}$ so $\det(H_f) = -72 < 0$. Therefore, $(-1,-2)$ is a $\boxed{\text{saddle point}}$.
- $H_f(1,-2) = \begin{bmatrix} 6 & 0 \\ 0 & 12 \end{bmatrix}$ so $\det(H_f) = 72 > 0$ and $f_{xx} = 6 > 0$. Therefore, $(1,-2)$ is a $\boxed{\text{local minimum}}$.
- $H_f(-1,2) = \begin{bmatrix} -6 & 0 \\ 0 & -12 \end{bmatrix}$ so $\det(H_f) = 72 > 0$ and $f_{xx} = -6 < 0$. Therefore, $(-1,2)$ is a $\boxed{\text{local maximum}}$.
- $H_f(1,2) = \begin{bmatrix} 6 & 0 \\ 0 & -12 \end{bmatrix}$ so $\det(H_f) = -72 < 0$. Therefore, $(1,2)$ is a $\boxed{\text{saddle point}}$.

[*Note:* Since the Hessian matrix is diagonal (it has zeros off of the diagonal), its diagonal entries are its eigenvalues. Thus we didn't really need the "full" second derivative test. For example, the eigenvalues of $H_f(1,-2)$ are 6 and 12. Since both are positive, we can "concave up" in "both" directions. Thus $(1,-2)$ is a relative minimum.]

7. (8 points) Let $z = \dfrac{y}{x^2}$. Use a differential (i.e. total derivative) to estimate the maximal **percent** error in z if x's measurement is off by at most 1% and y's measurement is off by at most 3%.

$z = yx^{-2}$ so $dz = z_x\,dx + z_y\,dy = -2yx^{-3}\,dx + x^{-2}\,dy$. Therefore, $\dfrac{dz}{z} = \dfrac{-2yx^{-3}\,dx + x^{-2}\,dy}{yx^{-2}} = \dfrac{-2yx^{-3}\,dx}{yx^{-2}} + \dfrac{x^{-2}\,dy}{yx^{-2}} = -2\dfrac{dx}{x} + \dfrac{dy}{y}$. The percent error in x is at most $\dfrac{dx}{x} = \pm 1\%$ and for y it's $\dfrac{dy}{y} = \pm 3\%$. Therefore, the percent error in z (i.e. dz/z) is at worst $-2(-1\%) + 3\% = \boxed{5\%}$.

8. (10 points) A Directed Problem. [Assume that the function $g(x,y)$ in parts (b) and (c) is differentiable.]

(a) Let $f(x,y,z) = e^{xyz} + 4x + yz^3$. Find the directional derivative of f at the point $(x,y,z) = (0,2,1)$ and in the same direction as $\mathbf{v} = \langle 1,-2,2 \rangle$.

First, we need to normalize our direction vector: $|\mathbf{v}| = \sqrt{1^2 + (-2)^2 + 2^2} = \sqrt{9} = 3$. Let $\mathbf{u} = \dfrac{\mathbf{v}}{|\mathbf{v}|} = \dfrac{1}{3}\langle 1,-2,2 \rangle$.

Next, we need to compute the gradient of f at the point $(x,y,z) = (0,2,1)$. $\nabla f = \langle f_x, f_y, f_z \rangle$
$= \langle yze^{xyz} + 4, xze^{xyz} + z^3, xye^{xyz} + 3yz^2 \rangle$. $\nabla f(0,2,1) = \langle 2(1)e^0 + 4, 0 + 1^3, 0 + 3(2)(1^2) \rangle = \langle 6,1,6 \rangle$.

$$D_{\mathbf{u}} f(0,2,1) = \nabla f(0,2,1) \bullet \mathbf{u} = \langle 6,1,6 \rangle \bullet \frac{1}{3}\langle 1,-2,2 \rangle = \frac{1}{3}(6 - 2 + 12) = \boxed{\frac{16}{3}}$$

(b) Suppose that $\nabla g(3,-1) = \langle 2,4\rangle$. What is the maximum possible value of $D_{\mathbf{u}}g(3,-1)$? Give a unit vector which causes this maximum to occur.

Since $D_{\mathbf{u}}g(3,-1) = \nabla g(3,-1) \bullet \mathbf{u} = |\nabla g(3,-1)|\,|\mathbf{u}|\cos(\theta) = |\nabla g(3,-1)|\cos(\theta)$ (θ is the angle between the gradient and $\mathbf{u}$) and $\cos(\theta)$ is bounded by ± 1, we get that the maximum value of the directional derivative (of g at the point $(3,-1)$) is $|\nabla g(3,-1)| = |\langle 2,4\rangle| = 2\sqrt{1^2+2^2} = \boxed{2\sqrt{5}}$.

This maximal value occurs when $\mathbf{u}$ points in the same direction as $\nabla g(3,-1)$ (i.e. $\theta = 0$). So $\mathbf{u} = \dfrac{\nabla g(3,-1)}{|\nabla g(3,-1)|} = \dfrac{1}{2\sqrt{5}}\langle 2,4\rangle = \boxed{\dfrac{1}{\sqrt{5}}\langle 1,2\rangle}$ maximizes the directional derivative of $g(x,y)$ at the point $(x,y) = (3,-1)$.

(c) Again, suppose $\nabla g(3,-1) = \langle 2,4\rangle$. Is it possible to find a unit vector $\mathbf{u}$ such that $D_{\mathbf{u}}g(3,-1) = -2$? Why or why not?

From part (b), we know that the maximal and minimal values of $D_{\mathbf{u}}g(3,-1)$ are $\pm 2\sqrt{5}$. In fact, $D_{\mathbf{u}}g(3,-1) = |\nabla g(3,-1)|\cos(\theta) = 2\sqrt{5}\cos(\theta)$. Thus by varying the unit vector $\mathbf{u}$, we can make $\cos(\theta)$ take on all values between -1 and 1, so since $-2\sqrt{5} \leq -2 \leq 2\sqrt{5}$, there is a unit vector $\mathbf{u}$ such that $D_{\mathbf{u}}g(3,-1) = -2$. **Yes.**

9. (10 points) Suppose $f(x,y)$ is a "nice" function (with continuous partials of all orders).

(a) $Q(x,y) = 1 + 3x + 6(y-5) + 2x^2 - 3x(y-5) + 4(y-5)^2$ is the quadratic approx. at $(x,y) = (0,5)$.

To find the gradient and Hessian matrix, just read off the appropriate coefficients and keep in mind that the x^2 term is actually $\dfrac{f_{xx}(0,5)}{2}x^2$ (so 2 should be doubled). Likewise, for the $(y-5)^2$ term. Recall that the mixed partials have been combined into a single term (from two terms), so the coefficient of $x(y-5)$ does *not* need to be doubled.

$$\nabla f(0,5) = \langle 3,6\rangle \qquad H_f(0,5) = \begin{bmatrix} 4 & -3 \\ -3 & 8 \end{bmatrix}$$

Is $(x,y) = (0,5)$ a critical point of $f(x,y)$? **NO**

If not, why not? If so, what kind (relative min, relative max, saddle point or not enough information)?

Since $\nabla f(0,5) \neq \langle 0,0\rangle$, $(0,5)$ is **not** a critical point.

(b) $Q(x,y) = 13 + 2(x+1)^2 - (x+1)(y-2) + 3(y-2)^2$ is the quadratic approx. at $(x,y) = (-1,2)$.

Same discussion as in part (a). Notice that $(x+1)$ and $(y-2)$ terms are missing, so their coefficients are 0.

$$\nabla f(-1,2) = \langle 0,0\rangle \qquad H_f(-1,2) = \begin{bmatrix} 4 & -1 \\ -1 & 6 \end{bmatrix}$$

Is $(x,y) = (-1,2)$ a critical point of $f(x,y)$? **YES**

If not, why not? If so, what kind (relative min, relative max, saddle point or not enough information)?

Since $\nabla f(-1,2) = \langle 0,0\rangle$, this is a critical point. Notice that $\det(H_f(-1,2)) = 4(6) - (-1)^2 = 23 > 0$ and $f_x x(-1,2) = 4 > 0$, so this is a **relative minimum**.

10. (12 points) Use the method of Lagrange multipliers to find the minimum and maximum values of $f(x,y) = xy$ constrained to $x^2 + 3y^2 = 18$.

Let $g(x,y) = x^2 + 3y^2$. Then $\nabla f(x,y) = \langle f_x, f_y\rangle = \langle y, x\rangle$ and $\nabla g(x,y) = \langle g_x, g_y\rangle = \langle 2x, 6y\rangle$.

Thus $\nabla f = \lambda \nabla g$ gives us $y = 2x\lambda$ and $x = 6y\lambda$. So we need to solve the system: $y = 2x\lambda$, $x = 6y\lambda$, and $x^2 + 3y^2 = 18$.

Let's symmetrize the first two equations (multiply the first by $3y$ and the second by x): $3y^2 = 6xy\lambda = x^2$. Thus $x^2 = 3y^2$. Plugging this into the constraint equation we get: $x^2 + x^2 = 18$ and $3y^2 + 3y^2 = 18$. Therefore, $x^2 = 9$ and $y^2 = 3$. Thus $x = \pm 3$ and $y = \pm\sqrt{3}$.

Finally, we plug in our points and find that: $f(3,\sqrt{3}) = f(-3,-\sqrt{3}) = 3\sqrt{3}$ and $f(3,-\sqrt{3}) = f(-3,\sqrt{3}) = -3\sqrt{3}$. So $f(x,y)$ constrained to $x^2 + 3y^2 = 18$ has a **maximum value of $3\sqrt{3}$** and a **minimum value of $-3\sqrt{3}$**.