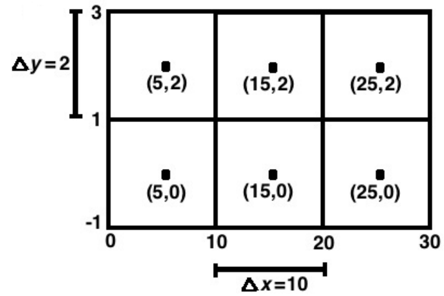
Math 2130-101 **Test #3** **July 27^{th}, 2015**

Name: Answer Key **Be sure to show your work!**

1. (10 points) Use a double Riemann sum to approximate $\displaystyle\iint_R \sqrt{x+y^2}\,dA$ where $R=[0,30]\times[-1,3]$.

Use midpoint rule and a 3×2 grid of rectangles (3 across and 2 up) to partition R. (Don't worry about simplifying.)



Draw a grid. $\Delta x = \dfrac{30-0}{3} = 10$ and $\Delta y = \dfrac{3-(-1)}{2} = 2$.

$$\iint_R \sqrt{x+y^2}\,dA \approx 10\cdot 2\Big(\sqrt{5+0^2}+\sqrt{15+0^2}+\sqrt{25+0^2} +\sqrt{5+2^2}+\sqrt{15+2^2}+\sqrt{25+2^2}\Big)$$

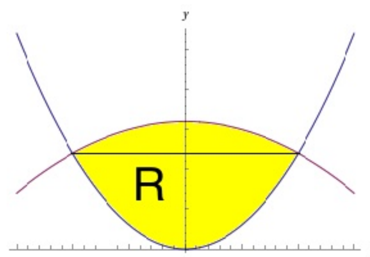
2. (10 points) Let R be the region bounded by $y=3x^2$ and $y=16-x^2$.

[Warning: One of the integrals below will have to be split into 2 pieces.]

(a) Sketch the region R.

(b) Set up the integral $\iint_R ye^x\,dA$ using the order of integration "$dy\,dx$". Don't evaluate the integral.

(c) Set up the integral $\iint_R ye^x\,dA$ using the order of integration "$dx\,dy$". Don't evaluate the integral.



This region is both x- and y-simple. However, as an x-simple region, we'll have to split it into 2 pieces. First, let's deal with the easier case (i.e. part (b)).
The top of this region is determined by $y=16-x^2$ (the parabola opening downward) and the bottom is determined by $y=3x^2$. Thus we have our y bounds. To get the x bounds we need to see where the top and bottom intersect: $16-x^2=y=3x^2$. Thus $16=4x^2$ and so $x^2=4$. Therefore, $x=\pm 2$.

$$\text{Part (b):}\qquad \iint_R ye^x\,dA = \int_{-2}^{2}\int_{3x^2}^{16-x^2} ye^x\,dy\,dx$$

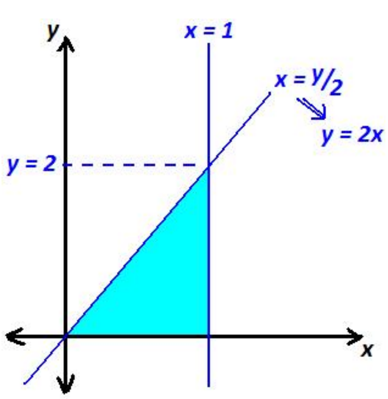
To treat this region as x-simple, we need to split up the region into a top part and bottom part (notice the line I've drawn in the plot of R). For the bottom half, $y=3x^2$ makes up both the left and right hand sides. Solving for x we get $x^2=y/3$ and so $x=\pm\sqrt{y/3}$. Likewise, $y=16-x^2$ makes up both the left and right hand sides of the top region: $x^2=16-y$ and so $x=\pm\sqrt{16-y}$.

Now we need to pin down the y bounds for both regions. The bottom region begins at $y=0$ (the vertex of the bottom parabola) and the top region ends at $y=16$ (the vertex of the top parabola). We already know that the parabolas meet when $x=\pm 2$ so that $y=3x^2=3(2^2)=12$ (or $y=16-x^2=16-2^2=12$).

$$\text{Part (c):}\qquad \iint_R ye^x\,dA = \int_0^{12}\int_{-\sqrt{y/3}}^{\sqrt{y/3}} ye^x\,dx\,dy \quad + \quad \int_{12}^{16}\int_{-\sqrt{16-y}}^{\sqrt{16-y}} ye^x\,dx\,dy$$

3. (10 points) Compute $\displaystyle\int_0^2\int_{y/2}^1 \cos(x^2)\,dx\,dy$.

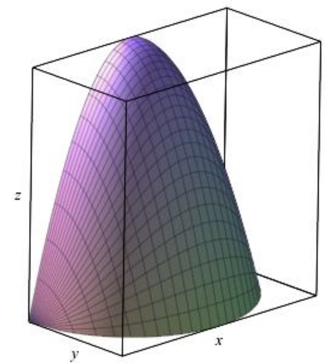
Hint: You cannot integrate $\int\cos(x^2)\,dx$ in terms of elementary functions.



Since it is impossible to integrate $\cos(x^2)$ with respect to x, we should reverse the order of integration. [*Note:* $\cos(x^2)\neq\cos^2(x)=(\cos(x))^2$ You cannot use a double angle identity.]
Reversing the order of integration: notice that y is bounded below by $y=0$ and above by $y=2x$ and also x ranges from $x=0$ to $x=1$. Thus we get:

$$\begin{aligned}\int_0^2\int_{y/2}^1\cos(x^2)\,dx\,dy &= \int_0^1\int_0^{2x}\cos(x^2)\,dy\,dx = \int_0^1 y\cos(x^2)\Big|_0^{2x}\,dx\\ &= \int_0^1 2x\cos(x^2)\,dx = \sin(x^2)\Big|_0^1 = \sin(1^2)-\sin(0^2)\\ &= \sin(1)\end{aligned}$$

4. (12 points) Let E be the region bounded by $z=0$ and $z=4-x^2-y^2$ and such that $y \geq 0$. A graph of this region is ever so kindly provided to the right. Set up integrals which compute the volume of E using the following orders of integration: [Do not evaluate these integrals.]



(a) $\int_?^? \int_?^? \int_?^? ???\, dz\, dy\, dx$

(b) $\int_?^? \int_?^? \int_?^? ???\, dx\, dz\, dy$

(c) $\int_?^? \int_?^? \int_?^? ???\, dy\, dx\, dz$

(d) Set up this integral in cylindrical coordinates.

Of course, $\iiint_E 1\, dV$ computes the volume of E. Now we need to find bounds for our iterated integrals.

The top of this region is obviously $z=4-x^2-y^2$ and the bottom is $z=0$. We should intersect these surfaces to find our y bounds. Doing so yields $0=z=4-x^2-y^2$ and so $x^2+y^2=4$. Now $y \geq 0$. Thus if we "squish out" the z-axis, we'll be left with a half disk in the xy-plane.

Solving for y in $x^2+y^2=4$ yields $y=\pm\sqrt{4-x^2}$. We only need the upper-half (the lower y bound is $y=0$ since $y \geq 0$). Finally the x bounds come from the extreme ends of our disk: $x=\pm 2$. Alternatively, intersecting $y=0$ and $y=\sqrt{4-x^2}$ yields $\sqrt{4-x^2}=0$ and so $4-x^2=0$ which again leads to $x=\pm 2$.

$$\text{Part (a):} \qquad \iiint_E 1\, dV = \int_{-2}^{2} \int_0^{\sqrt{4-x^2}} \int_0^{4-x^2-y^2} 1\, dz\, dy\, dx$$

Next, treating E as an x-simple region, we need to determine E's "back" and "front". Looking at the graph both the back and front of this region are determined by the paraboloid. Thus solving $z=4-x^2-y^2$ for x yields $x=\pm\sqrt{4-y^2-z}$ (the x bounds).

With x bounds in hand, we need to find the z bounds (since we are setting up the order: $dx\, dz\, dy$). Squishing out the x-axis leaves us with a half-parabola kind of region. This is bounded on the bottom by $z=0$, on the left by $y=0$, and on the top and right by where our x bounds intersect: $-\sqrt{4-y^2-z}=\sqrt{4-y^2-z}$ and so $2\sqrt{4-y^2-z}=0$ and so $4-y^2-z=0$ and thus $z=4-y^2$ (i.e. $x=0$ in $z=4-x^2-y^2$). So we have our z-bounds: $0 \leq z \leq 4-y^2$.

This leaves our y-bounds. These come from intersecting the z-bounds: $0=z=4-y^2$ and so $y^2=4$ and so $y=\pm 2$. Remember that $y \geq 0$ so we should forget the $y=-2$ solution. Our y-bounds are: $0 \leq y \leq 2$. If we remembered our work from part (a) where we ended up with the upper-half of a disk of radius 2, we could have just written down this set of bounds without any work!

$$\text{Part (b):} \qquad \iiint_E 1\, dV = \int_0^2 \int_0^{4-y^2} \int_{-\sqrt{4-y^2-z}}^{\sqrt{4-y^2-z}} 1\, dx\, dz\, dy$$

On to the next order of integration. This time y comes first. We need to determine the "left" and "right" sides of our region. Looking at the graph, the left side is determined by $y=0$ and the right side is determined by the paraboloid: $z=4-x^2-y^2$. We need to solve this for y: $y=\pm\sqrt{4-x^2-z}$. Remember that $y \geq 0$ so we should forget the negative solution. Our y bounds are: $0 \leq y \leq \sqrt{4-x^2-z}$.

Squishing our the y-axis leaves us with a parabolic region in the xz-plane. This is bounded below by $z=0$ and above by where the paraboloid $z=4-x^2-y^2$ intersects the xz-plane (i.e. $y=0$) and so we get $z=4-x^2$. The left and right sides of our region in the xz-plane are then given by this parabola: $z=4-x^2$. Solving for x we get $x=\pm\sqrt{4-z}$.

Finally, our z-bounds come from intersecting our x bounds: $-\sqrt{4-z}=\sqrt{4-z}$ so $2\sqrt{4-z}=0$ and so $4-z=0$. Thus $z=4$ (or just notice that this is the vertex of the paraboloid).

$$\text{Part (c):} \qquad \iiint_E 1\, dV = \int_0^4 \int_{-\sqrt{4-z}}^{\sqrt{4-z}} \int_0^{\sqrt{4-x^2-z}} 1\, dy\, dx\, dz$$

Our last task is to switch to cylindrical coordinates. This is easy. We already have z bounds: $0 \leq z \leq 4-x^2-y^2$. In cylindrical coordinates these become: $0 \leq z \leq 4-r^2$.

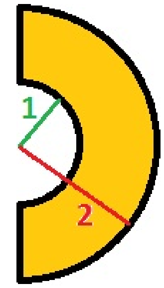
Squishing out z we get a upper-half disk: $x^2+y^2 \leq 4$, $y \geq 0$. This translates to $0 \leq r \leq 2$ (from $x^2+y^2=r^2=4$) and $0 \leq \theta \leq \pi$ (from $y \geq 0$). Finally, don't forget the Jacobian!

$$\text{Part (d):} \qquad \iiint_E 1\, dV = \int_0^{\pi} \int_0^2 \int_0^{4-r^2} r\, dz\, dr\, d\theta$$

We aren't asked to compute the volume, but if we were, this last integral is the easiest to evaluate:

$$= \int_0^{\pi} d\theta \int_0^2 r(4-r^2)\, dr = \pi \int_0^2 4r - r^3\, dr = \pi \left(2r^2 - \frac{1}{4}r^4\right)\Bigg|_0^2 = 4\pi$$

5. (10 points) Compute $\iint_R 2e^{x^2+y^2}\,dA$ where R is the part of the annulus $1 \le x^2+y^2 \le 4$ where $x \ge 0$.

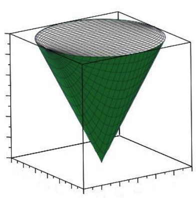


R is an annular region (as pictured to the right). We see x^2+y^2 appear several times, let's switch to polar coordinates.
In polar coordinates $1 \le x^2+y^2 \le 4$ becomes $1 \le r^2 \le 4$ and so $1 \le r \le 2$. The restriction $x \ge 0$ tells us that $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. Finally, $e^{x^2+y^2} = e^{r^2}$ and don't forget the Jacobian! [$\int 2re^{r^2}\,dr = e^{r^2} + C$ by a u-sub: $u = r^2$.]

$$\iint_R 2e^{x^2+y^2}\,dA = \int_{-\pi/2}^{\pi/2}\int_1^2 2e^{r^2} r\,dr\,d\theta = \int_{-\pi/2}^{\pi/2} d\theta \int_1^2 2r\,e^{r^2}\,dr = \pi e^{r^2}\Big|_1^2 = \pi(e^4 - e)$$

6. (12 points) Consider the region E above the cone $z = \sqrt{x^2+y^2}$ and below $z = 1$. Sketch this region then find its centroid. Recall that...

$$(\bar{x},\bar{y},\bar{z}) = \frac{1}{m}(M_{yz}, M_{xz}, M_{xy}) \qquad m = \iiint_E 1\,dV \qquad M_{yz} = \iiint_E x\,dV \qquad M_{xz} = \iiint_E y\,dV \qquad M_{xy} = \iiint_E z\,dV$$



This is a flat topped conical region. We get $\bar{x} = \bar{y} = 0$ by symmetry. We need to compute m and M_{xy} so that we can find $\bar{z}$.
This region is best dealt with in cylindrical coordinates. We have $z = \sqrt{x^2+y^2} = r$ and $z = 1$ (bottom and top z bounds). Intersecting these we get $r = z = 1$ so $0 \le r \le 1$. Finally, $0 \le \theta \le 2\pi$. [If we squish out the z-axis, we would be left with a disk of radius 1 in the xy-plane.] Don't forget the Jacobian!

$$m = \iiint_E 1\,dV = \int_0^{2\pi}\int_0^1\int_r^1 1\cdot r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^1\int_r^1 r\,dz\,dr = 2\pi\int_0^1 rz\Big|_r^1 = 2\pi\int_0^1 r - r^2\,dr = 2\pi\left(\frac{1}{2}-\frac{1}{3}\right) = \frac{\pi}{3}$$

$$M_{xy} = \iiint_E z\,dV = \int_0^{2\pi}\int_0^1\int_r^1 z\cdot r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^1\int_r^1 rz\,dz\,dr = 2\pi\int_0^1 \frac{r}{2}z^2\Big|_r^1 = 2\pi\int_0^1 \frac{r}{2} - \frac{r^3}{2}\,dr = 2\pi\left(\frac{1}{4}-\frac{1}{8}\right) = \frac{\pi}{4}$$

$$\text{Centroid:} \qquad (\bar{x},\bar{y},\bar{z}) = \left(0, 0, \frac{\pi/4}{\pi/3}\right) = \left(0,0,\frac{3}{4}\right)$$

7. (12 points) Compute $\iint_R \frac{-3x+y}{x+y}\,dA$ where R is the region bounded by $y = 3x+1$, $y = 3x+2$, $y = -x+1$, and $y = -x+3$. Use a (natural) change of coordinates which simplifies the region R and... don't forget the Jacobian!

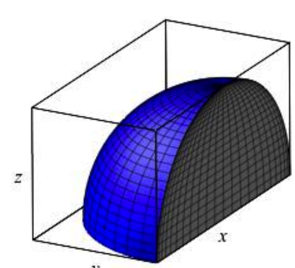
Notice that our region is bounded by $-3x+y = 1$, $-3x+y = 2$, $x+y = 1$, and $x+y = 3$. Using the natural change of coordinates: $u = -3x+y$ and $v = x+y$, we get that $u = 1$, $u = 2$, $v = 1$, and $v = 3$. The function to be integrated is just $\frac{-3x+y}{x+y} = \frac{u}{v}$. After we compute the Jacobian, we'll be ready to integrate.

$$J^{-1} = \frac{\partial(u,v)}{\partial(x,y)} = \det\begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} = \det\begin{bmatrix} -3 & 1 \\ 1 & 1 \end{bmatrix} = (-3)1 - 1(1) = -4$$

Notice that we have computed partials of *new* variables in terms of *old* variables. This is the *inverse* Jacobian. So $J = 1/J^{-1} = -1/4$. [Alternatively, we could have solve $u = -3x+y$ and $v = x+y$ for x and y. Subtracting gives: $u - v = -4x$ so $x = (-1/4)u + (1/4)v$. Multiplying the second equation by 3 and adding gets rid of x: $u + 3v = 4y$ so $y = (1/4)u + (3/4)v$. The Jacobian then is $\det\begin{bmatrix} -1/4 & 1/4 \\ 1/4 & 3/4 \end{bmatrix} = -1/4$.]

$$\iint_R \frac{-3x+y}{x+y}\,dA = \int_1^3\int_1^2 \frac{u}{v}\cdot\left|-\frac{1}{4}\right|\,du\,dv = \frac{1}{4}\int_1^3 \frac{1}{v}\,dv\int_1^2 u\,du = \frac{1}{4}\left(\ln(3)-\ln(1)\right)\left(\frac{1}{2}2^2 - \frac{1}{2}1^2\right) = \frac{3\ln(3)}{8}$$

8. (12 points) Consider the integral: $I = \int_{-5}^{5}\int_{-\sqrt{25-x^2}}^{0}\int_0^{\sqrt{25-x^2-y^2}} \sin(x^2+y^2+z^2)\,dz\,dy\,dx$. Do **not** evaluate these integrals.



(a) Rewrite I in the following order of integration: $\iiint \quad dy\,dx\,dz$.

(b) Rewrite I in terms of cylindrical coordinates.

(c) Rewrite I in terms of spherical coordinates.

Our bounds translate to $0 \le z \le \sqrt{25-x^2-y^2}$, $-\sqrt{25-x^2} \le y \le 0$, and $-5 \le x \le 5$. The first set of bounds indicate the upper half of a sphere of radius 5. The second set of bounds indicate we should chop the upper half of the sphere in half again keeping the "right" part (negative y part). This yields the picture above.

Our sphere is $x^2+y^2+z^2=25$. Solving for y yields $y = \pm\sqrt{25-x^2-z^2}$. But $y \le 0$ so we get $-\sqrt{25-x^2-z^2} \le y \le 0$. Next, after squishing out the y-axis, we are left with the upper half of a disk in the xz-plane. We solve $x^2+z^2=25$ for x: $x = \pm\sqrt{25-z^2}$ (the left and right sides of our half disk). Finally, $0 \le z \le 5$.

$$\text{Part (a):} \quad \int_0^5 \int_{-\sqrt{25-z^2}}^{\sqrt{25-z^2}} \int_{-\sqrt{25-x^2-z^2}}^{0} \sin(x^2+y^2+z^2)\, dy\, dx\, dz$$

Next, in cylindrical coordinates, $0 \le z \le \sqrt{25-x^2-y^2} = \sqrt{25-r^2}$. Squishing out the z-axis yields the lower half of a disk of radius 5 in the xy-plane. So $0 \le r \le 5$ and $\pi \le \theta \le 2\pi$. Finally, $\sin(x^2+y^2+z^2) = \sin(r^2+z^2)$ and don't forget the Jacobian!

$$\text{Part (b):} \quad \int_\pi^{2\pi} \int_0^5 \int_0^{\sqrt{25-r^2}} \sin(r^2+z^2)\, r\, dz\, dr\, d\theta$$

Finally, in spherical coordinates this sphere is $\rho^2 = 25$ so $\rho = 5$. We get $0 \le \rho \le 5$. θ is the same as in cylindrical coordinates. Since we are restricted to the upper half of the sphere, $0 \le \varphi \le \pi/2$. Our function becomes $\sin(x^2+y^2+z^2) = \sin(\rho^2)$ and don't forget the Jacobian!

$$\text{Part (c):} \quad \int_\pi^{2\pi} \int_0^{\pi/2} \int_0^5 \sin(\rho^2)\, \rho^2 \sin(\varphi)\, d\rho\, d\varphi\, d\theta$$

9. (12 points) Let E be the region bounded below by the cone $z = \sqrt{3x^2+3y^2}$ $(= \sqrt{3} \cdot \sqrt{x^2+y^2})$ and above by the hemisphere $z = \sqrt{4-x^2-y^2}$.

(a) Write down an integral which computes the volume of E in cylindrical coordinates. Do not evaluate this integral.

This is an "ice cream cone" shaped region. This is much like the cone in problem #6 except that we have a spherical top instead of a flat top.

The bottom of this region is determined by $z = \sqrt{3x^2+3y^2} = \sqrt{3r^2} = \sqrt{3}\,r$. The top of this region is determined by $z = \sqrt{4-x^2-y^2} = \sqrt{4-r^2}$. Thus we have our z bounds.

To find r bounds we should intersect the top and bottom surfaces: $\sqrt{3}\,r = z = \sqrt{4-r^2}$. Squaring both sides yields $3r^2 = 4-r^2$ so that $4r^2 = 4$. Thus $r=1$. So if we squish out the z-axis, we'll be left with a disk of radius 1. Don't forget the Jacobian!

$$\text{Volume}(E) = \iiint_E 1\, dV = \int_0^{2\pi} \int_0^1 \int_{\sqrt{3}\,r}^{\sqrt{4-r^2}} 1 \cdot r\, dz\, dr\, d\theta$$

(b) Write down an integral which computes the volume of E in spherical coordinates. Do not evaluate this integral.

Imagine fixing a particular θ and φ, so we get a ray emanating from the origin. This will pass through E and exit when we get to the sphere. This sphere has equation: $z = \sqrt{4-x^2-y^2}$ so that $x^2+y^2+z^2=4$ and so $\rho^2=4$. This gives us our ρ bounds $0 \le \rho \le 2$.

θ remains the same as in part (a), so at this point we just lack our φ bounds. These must be determined by the cone itself. The cone's equation is $z = \sqrt{3x^2+3y^2}$. Either using the regular change of coordinate formulas and some algebra or recalling that $r = \rho\sin(\varphi)$, we get that $\rho\cos(\varphi) = \sqrt{3}\,\rho\sin(\varphi)$. Therefore, $\cos(\varphi) = \sqrt{3}\sin(\varphi)$ which implies that $\tan(\varphi) = \dfrac{\sin(\varphi)}{\cos(\varphi)} = \dfrac{1}{\sqrt{3}}$. The 30°-60°-90° triangle has sides of length 1, $\sqrt{3}$, and 2. In fact, $\tan(30°) = \tan(\pi/6) = \dfrac{1}{\sqrt{3}}$. Thus the cone's equation is just $\varphi = \pi/6$. So φ sweeps out from the z-axis until we hit the cone. This means that $0 \le \varphi \le \pi/6$.

We've found all of our bounds so we can write down our integral. One last time... don't forget the Jacobian!

$$\text{Volume}(E) = \iiint_E 1\, dV = \int_0^{2\pi} \int_0^{\pi/6} \int_0^2 1 \cdot \rho^2 \sin(\varphi)\, d\rho\, d\varphi\, d\theta$$

This last integral is fairly easy to evaluate. We weren't asked to, but if we did evaluate it, we would find that the volume is $\dfrac{2-\sqrt{3}}{3} \cdot 8\pi$.