Math 2130 **Test #1** **July 11$^{\text{th}}$, 2016**

Name: ANSWER KEY **Be sure to show your work!**

1. (20 points) Vector Basics: Let $\mathbf{v} = \langle 3, -1, 2\rangle$ and $\mathbf{w} = \langle 2, 1, -1\rangle$.

(a) Find a **unit** vector perpendicular to both $\mathbf{v}$ and $\mathbf{w}$.

First, the cross product $\mathbf{v} \times \mathbf{w}$ will give us a vector perpendicular to both $\mathbf{v}$ and $\mathbf{w}$.

$$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -1 & 2 \\ 2 & 1 & -1 \end{vmatrix} = \begin{vmatrix} -1 & 2 \\ 1 & -1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 3 & 2 \\ 2 & -1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 3 & -1 \\ 2 & 1 \end{vmatrix} = -\mathbf{i} + 7\mathbf{j} + 5\mathbf{k}$$

But we want a **unit** vector that's perpendicular to $\mathbf{v}$ and $\mathbf{w}$, so we need to normalize the cross product. Notice that $|\mathbf{v} \times \mathbf{w}| = |\langle -1, 7, 5\rangle| = \sqrt{(-1)^2 + 7^2 + 5^2} = \sqrt{75} = 5\sqrt{3}$.

Answer: $\boxed{\dfrac{1}{5\sqrt{3}}\langle -1, 7, 5\rangle}$ is a unit vector which is perpendicular to both $\mathbf{v}$ and $\mathbf{w}$.

(b) Compute $\text{proj}_{\mathbf{v}}(\mathbf{w}) = \dfrac{\mathbf{v} \bullet \mathbf{w}}{|\mathbf{v}|^2}\mathbf{v} = \dfrac{3(2) + (-1)1 + 2(-1)}{3^2 + (-1)^2 + 2^2}\langle 3, -1, 2\rangle = \boxed{\dfrac{3}{14}\langle 3, -1, 2\rangle}$

(c) Find the angle between $\mathbf{v}$ and $\mathbf{w}$ (don't worry about evaluating inverse trig. functions).

Recall that $\mathbf{v} \bullet \mathbf{w} = |\mathbf{v}||\mathbf{w}|\cos(\theta)$ where θ is the angle between $\mathbf{v}$ and $\mathbf{w}$. So noting that $\mathbf{v} \bullet \mathbf{w} = 3$, $|\mathbf{v}| = \sqrt{14}$, $|\mathbf{w}| = \sqrt{2^2 + 1^2 + (-1)^2} = \sqrt{6}$, and solving for θ we get:

$$\theta = \arccos\left(\frac{\mathbf{v} \bullet \mathbf{w}}{|\mathbf{v}||\mathbf{w}|}\right) = \arccos\left(\frac{3}{\sqrt{14}\sqrt{6}}\right) = \boxed{\arccos\left(\frac{3}{2\sqrt{21}}\right)}$$

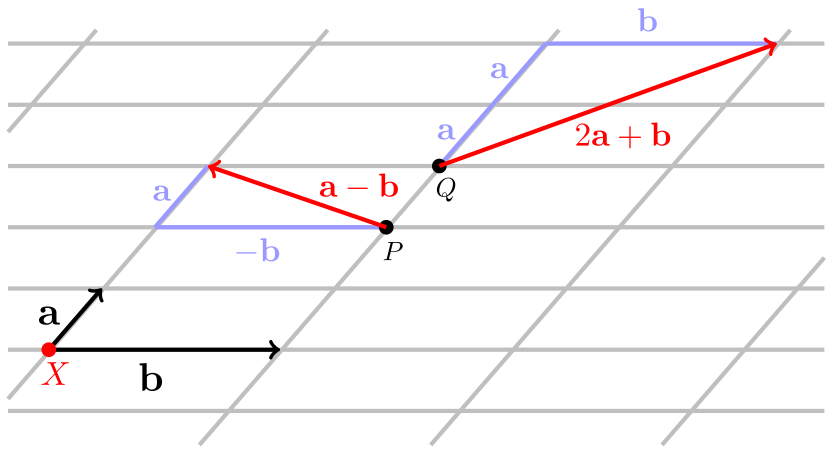
Is this angle... **right,** $\boxed{\textbf{acute}}$ (because $\mathbf{v} \bullet \mathbf{w} = 3 > 0$)**,** or **obtuse** ? (Circle your answer.)

(d) Fill in the blanks ($\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are vectors)...

(i) $|\mathbf{a} \times \mathbf{b}|$ computes the **area** of the **parallelogram** spanned by $\mathbf{a}$ and $\mathbf{b}$.

(ii) $|\mathbf{a} \bullet (\mathbf{b} \times \mathbf{c})|$ computes the **volume** of the **parallelepiped** spanned by $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$.

(e) The vectors $\mathbf{a}$ and $\mathbf{b}$ are shown to the right. They are based at the point X. Sketch the vector $\mathbf{a} - \mathbf{b}$ based at the point P and sketch the vector $2\mathbf{a} + \mathbf{b}$ based at the point Q.



2. (10 points) Let ℓ_1 be parametrized by $\mathbf{r}_1(t) = \langle 2t+1, -t+3, t\rangle$ and let ℓ_2 be the line which passes through the points $P = (2,1,-1)$ and $Q = (-2,3,-3)$. Determine if ℓ_1 and ℓ_2 are... (circle the correct answer)

the same, $\boxed{\text{parallel (but not the same)}}$, intersecting, or skew.

Frist, let's parameterize the second line: $\mathbf{r}_2(t) = P + \overrightarrow{PQ}\,t = P + (Q-P)t = \langle 2,1,-1\rangle + \langle -4,2,-2\rangle t$.

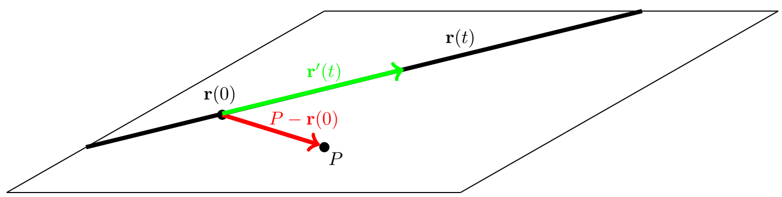
To see if these lines are parallel or not, we can look at their direction vectors: $\mathbf{r}_1'(t) = \langle 2,-1,1\rangle$ and $\mathbf{r}_2'(t) = \langle -4,2,-2\rangle$. Notice that the second vector is just -2 times the first one. Thus our lines are either the same line or distinct parallel lines.

Now we need to see if they intersect. Let's try to solve the equation $\mathbf{r}_1(s) = \mathbf{r}_2(t)$. This yields: $2s+1 = 2-4t$, $-s+3 = 1+2t$, and $s = -1-2t$. Plugging the third equation into the second gives: $-(-1-2t)+3 = 1+2t$ and so $2t+4 = 1+2t$. This means $4 = 1$. Thus this system of equations is inconsistent and so the lines do not intersect (they are not the same line).

3. (13 points) Plane old geometry.

(a) Find a (scalar) equation for the plane that containing the line $\mathbf{r}(t) = \langle 1+2t, -1-t, -2t\rangle$ and the point $P = (1,0,-2)$.

I've drawn a diagram to the right to help visualize what is going on.
First, notice that $\mathbf{r}'(t) = \langle 2,-1,-2\rangle$ is parallel to the line and thus parallel to the plane that the line lies in. Next, if we create a vector pointing from any point on the line to the point P, it will be parallel to our plane as well. Let's go from $\mathbf{r}(0)$ to P: $P - \mathbf{r}(0) = \langle 0,1,-2\rangle$.



Now we have two vectors which are parallel to the plane (but not each other). If we cross product them, we'll have a normal for our plane.

$$\mathbf{n} = \langle 2,-1,-2\rangle \times \langle 0,1,-2\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \end{vmatrix} = \begin{vmatrix} -1 & -2 \\ 1 & -2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 2 & -2 \\ 0 & -2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 2 & -1 \\ 0 & 1 \end{vmatrix}\mathbf{k} = \langle 4,4,2\rangle$$

The plane passes through the point $P = (1,0,-2)$ so we get $4(x-1)+4(y-0)+2(z-(-2)) = 0$ which simplifies to $4x+4y+2z = 0$ or $\boxed{2x+2y+z=0}$.

(b) Consider the two planes: $x+2y+2z+4 = 0$ and $2x-y-2z+5 = 0$. If these are parallel planes explain why they are parallel. If these planes intersect, find the angle between these planes (don't worry about evaluating inverse trig. functions).

Notice that $x+2y+2z+4 = 0$ has a normal vector $\mathbf{n}_1 = \langle 1,2,2\rangle$ and $2x-y-2z+5 = 0$ has a normal vector $\mathbf{n}_2 = \langle 2,-1,-2\rangle$. Since these vectors are not parallel (they are obviously not multiples of each other), we must have $\boxed{\text{intersecting planes}}$.

Recall that the angle between the normal vectors (as long as it's not obtuse) is the angle between the planes. Therefore, the angle between our planes is (the absolute value encasing the dot product is to force an acute angle)...

$$\arccos\left(\frac{|\mathbf{n}_1 \bullet \mathbf{n}_2|}{|\mathbf{n}_1|\,|\mathbf{n}_2|}\right) = \arccos\left(\frac{|1(2)+2(-1)+2(-2)|}{\sqrt{1^2+2^2+2^2}\cdot\sqrt{2^2+(-1)^2+(-2)^2}}\right) = \arccos\left(\frac{|-4|}{3\cdot 3}\right) = \boxed{\arccos\left(\frac{4}{9}\right)}$$

4. (8 points) Find the area of the triangle with vertices $A = (1,0,2)$, $B = (2,2,5)$, and $C = (1,3,3)$.

We note that the parallelogram spanned by $\overrightarrow{AB}$ and $\overrightarrow{AC}$ is twice the area of the triangle ΔABC. Since the length of a cross product computes the area of the parallelogram spanned by the vectors being crossed, we get that the area of ΔABC is $\frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}|$.

$$\overrightarrow{AB} \times \overrightarrow{AC} = (B-A)\times(C-A) = \langle 1,2,3\rangle \times \langle 0,3,1\rangle = \langle -7,-1,3\rangle \implies \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \frac{1}{2}|\langle -7,-1,3\rangle| = \frac{\sqrt{59}}{2}$$

5. (8 points) Compute the curvature of $\mathbf{r}(t) = \langle t^2, \sin(t), e^{2t}\rangle$ **[Don't attempt simplifying!]**

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \langle 2t, \cos(t), 2e^{2t}\rangle \times \langle t, -\sin(t), 4e^{2t}\rangle = \langle 4\cos(t)e^{2t} + 2e^{2t}\sin(t), -(8te^{2t} - 4e^{2t}), -2t\sin(t) - 2\cos(t)\rangle$$

$$\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \boxed{\frac{\sqrt{4e^{4t}(2\cos(t)+\sin(t))^2 + 16e^{4t}(2t-1)^2 + 4(t\sin(t)+\cos(t))^2}}{(4t^2+\cos^2(t)+4e^{4t})^{3/2}}}$$

6. (13 points) Consider the circle C: $(x-a)^2+(y-b)^2=R^2$ (where $R>0$).

(a) Parameterize C and compute its arc length using your parameterization.

$$\text{The standard parameterization: } \boxed{\mathbf{r}(t)=\langle R\cos(t)+a, R\sin(t)+b\rangle \quad \text{where} \quad 0\le t\le 2\pi}$$

$\mathbf{r}'(t)=\langle -R\sin(t), R\cos(t)\rangle$ so that $|\mathbf{r}'(t)|=\sqrt{R^2\sin^2(t)+R^2\cos^2(t)}=\sqrt{R^2(\cos^2(t)+\sin^2(t))}=\sqrt{R^2}=R.$

$$\text{Arc Length} = \int_C 1\,ds = \int_0^{2\pi} |\mathbf{r}'(t)|\,dt = \int_0^{2\pi} R\,dt = \boxed{2\pi R}$$

(b) Compute the curvature of C using your parameterization.

We may as well use the formula $\kappa(t)=|\mathbf{T}'(t)|/|\mathbf{r}'(t)|$ since we've almost computed $\mathbf{T}(t)$ anyway.

$$\mathbf{T}(t)=\frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}=\frac{1}{R}\langle -R\sin(t), R\cos(t)\rangle=\langle -\sin(t),\cos(t)\rangle \implies \mathbf{T}'(t)=\langle -\cos(t), -\sin(t)\rangle \implies |\mathbf{T}'(t)|=1$$

Therefore, $\kappa(t)=\dfrac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}=\boxed{\dfrac{1}{R}}$.

7. (16 points) Consider the curve parameterized by $\mathbf{r}(t)=\langle 3\sin(t), 4t, -3\cos(t)\rangle$ where $0\le t\le 2\pi$.

(a) Compute the centroid of this curve.

To compute the necessary line integrals we need to know the arc length element: $ds=|\mathbf{r}'(t)|\,dt$. $\mathbf{r}'(t)=\langle 3\cos(t), 4, 3\sin(t)\rangle$ and so $|\mathbf{r}'(t)|=\sqrt{9\cos^2(t)+16+9\sin^2(t)}=\sqrt{9(\cos^2(t)+\sin^2(t))+16}=\sqrt{9+16}=5.$

$$m=\int_C 1\,ds=\int_0^{2\pi}|\mathbf{r}'(t)|\,dt=\int_0^{2\pi}5\,dt=10\pi \qquad M_{yz}=\int_C x\,ds=\int_0^{2\pi}(3\sin(t))5\,dt=0$$

$$M_{xz}=\int_C y\,ds=\int_0^{2\pi}(4t)5\,dt=10t^2\Big|_0^{2\pi}=40\pi^2 \qquad M_{xy}=\int_C z\,ds=\int_0^{2\pi}(-3\cos(t))5\,dt=0$$

The centroid of C is $(\bar{x},\bar{y},\bar{z})=\left(\dfrac{M_{yz}}{m},\dfrac{M_{xz}}{m},\dfrac{M_{xy}}{m}\right)=\left(0,\dfrac{40\pi^2}{10\pi},0\right)=\boxed{(0,4\pi,0)}$.

(b) Find the TNB-frame for $\mathbf{r}(t)$.

Putting together calculations from part (a) we get that $\mathbf{T}(t)=\dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}=\boxed{\dfrac{\langle 3\cos(t),4,3\sin(t)\rangle}{5}=\mathbf{T}(t)}$ Next, $\mathbf{T}'(t)=\dfrac{1}{5}\langle -3\sin(t),0,3\cos(t)\rangle=\dfrac{3}{5}\langle -\sin(t),0,\cos(t)\rangle$ and so $|\mathbf{T}'(t)|=\dfrac{3}{5}\sqrt{\sin^2(t)+0+\cos^2(t)}=\dfrac{3}{5}$. Thus $\mathbf{N}(t)=\dfrac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}=\dfrac{\frac{3}{5}\langle -\sin(t),0,\cos(t)\rangle}{\frac{3}{5}}=\boxed{\langle -\sin(t),0,\cos(t)\rangle=\mathbf{N}(t)}$. Finally, $\mathbf{B}(t)=\mathbf{T}(t)\times\mathbf{N}(t)$
$=\dfrac{1}{5}\langle 3\cos(t),4,3\sin(t)\rangle\times\langle -\sin(t),0,\cos(t)\rangle=\dfrac{1}{5}\langle 4\cos(t),-(3\cos^2(t)+3\sin^2(t)),4\sin(t)\rangle=\boxed{\dfrac{1}{5}\langle 4\cos(t),-3,4\sin(t)\rangle=\mathbf{B}(t)}$.

Does this curve lie in a plane? Why or why not?

$\boxed{\textbf{No}}$, C does not lie in a plane. Planar curves have constant binormals. Obviously, C's binormal, $\mathbf{B}(t)$, is not constant. Therefore, C cannot be a planar curve.

8. (12 points) Choose **ONE** of the following: [In both cases, drawing a good explanatory picture will earn you some partial credit – but for full credit you need more.]

I. Let $\mathbf{a}$ and $\mathbf{b}$ be vectors. Simplify $(2\mathbf{a}-\mathbf{b})\bullet(\mathbf{a}\times\mathbf{b})$. Explain your answer geometrically.

Recall that the triple scalar product $\mathbf{c}\bullet(\mathbf{a}\times\mathbf{b})$ computes $\pm$ the volume of the parallelepiped spanned by $\mathbf{c}$, $\mathbf{a}$, and $\mathbf{b}$. Notice that $\mathbf{c}=2\mathbf{a}-\mathbf{b}$ lies in the plane spanned by $\mathbf{a}$ and $\mathbf{b}$. Since these vectors are coplanar, the parallelepiped has no volume. This means that $\boxed{(2\mathbf{a}-\mathbf{b})\bullet(\mathbf{a}\times\mathbf{b})=0 \text{ because the vectors are coplanar.}}$

We can also get this result more algebraically: $(2\,\mathbf{a}-\mathbf{b})\bullet(\mathbf{a}\times\mathbf{b}) = 2\,\mathbf{a}\bullet(\mathbf{a}\times\mathbf{b})-\mathbf{b}\bullet(\mathbf{a}\times\mathbf{b}) = 2(0)-0=0$ [the dot products are 0 because $\mathbf{a}\times\mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.]

II. Let C be the curve in the plane defined by $y=f(x)$ (where f is a function with at least two continuous derivatives). Use the special formula for curvature to explain why $\kappa(x)=0$ exactly when C is a line (or part of a line).

Suppose that C defined by $y=f(x)$ is a line (or part of a line). This means that $f(x)=mx+b$ for some constants m and b. Therefore, $f'(x)=m$ and so $f''(x)=0$. This means that $\kappa(x)=\dfrac{|f''(x)|}{(1+(f'(x))^2)^{3/2}}=\dfrac{0}{(1+m^2)^{3/2}}=0$.

By the way, conversely, if $\kappa(x)=0$, we must have $f''(x)=0$ and so $f'(x)=m$ for some constant m. Thus $f(x)=mx+b$ for some constant b and, poof!, $y=f(x)$ is a line (or piece of a line).