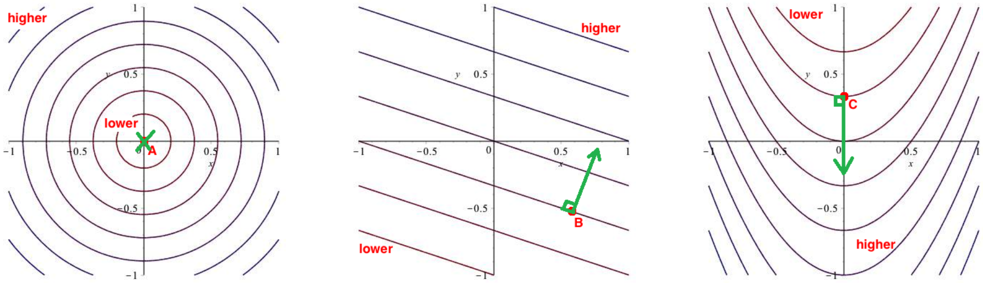
Math 2130 **Test #2** **July 20th, 2016**

Name: ANSWER KEY **Be sure to show your work!**

1. (10 points) Three level curve plots are shown below. I have labeled the levels so you know which curves are higher and which are lower.



$\boxed{5\sqrt{x^2+y^2}}$ $\boxed{x+3y-1}$ $\boxed{4x^2-3y}$

(a) The plots above correspond to 3 of the functions listed here: $f(x,y)=9-x^2-y^2$, $f(x,y)=x+3y-1$, $f(x,y)=5\sqrt{x^2+y^2}$, $f(x,y)=4x^2-3y$, and $f(x,y)=4y^2-3x$. Write the correct formula below each plot.

The level curves of $f(x,y)=9-x^2-y^2$ and $f(x,y)=5\sqrt{x^2+y^2}$ are $9-x^2-y^2=c$ thus $x^2+y^2=9-c$ and $5\sqrt{x^2+y^2}=c$ thus $x^2+y^2=c^2/25$ respectively. The other functions don't have circles for level curves, so the first plot must go with one of these functions. Notice that as $z=c$ gets higher $x^2+y^2=9-c$ gets smaller wheres $x^2+y^2=c^2/25$ gets bigger. Since the plot shows bigger circles go with higher levels, $f(x,y)=5\sqrt{x^2+y^2}$ must be its formula. By the way, the graph of $f(x,y)=5\sqrt{x^2+y^2}$ is a cone. This fits perfectly.

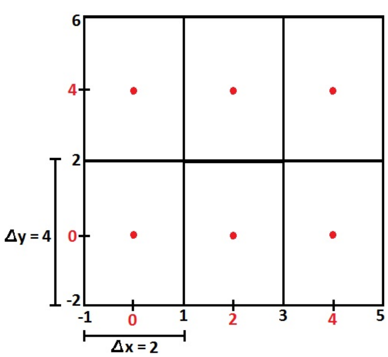
Next, the middle plot has lines as level curves. The only function matching this is $f(x,y)=x+3y-1$ (with level curves $x+3y-1=c$). This function's graph is a plane.

Finally, both $f(x,y)=4x^2-3y$ and $f(x,y)=4y^2-3x$ have parabolic level curves. Notice that $4x^2-3y=c$ means $y=\frac{4}{3}x^2-\frac{c}{3}$ whereas $4y^2-3x=c$ means $x=\frac{4}{3}y^2-\frac{c}{3}$. The first function's level curves are parabolas opening upward. The second function's level curves are parabolas opening to the right. Thus the third plot must go with $f(x,y)=4x^2-3y$.

(b) Sketch a gradient vector at the points A, B, and C. If the vector is **0**, draw an "X" on the point.
[Don't worry about having the correct length. I'm just looking for the correct direction.]

Recall that gradient vectors are perpendicular to level curves and point toward higher levels. The gradient in the first plot is the zero vector (well, technically the gradient is undefined here: $f(x,y)=5\sqrt{x^2+y^2}$ implies $\nabla f(x,y)=\langle 5x(x^2+y^2)^{-1/2}, 5y(x^2+y^2)^{-1/2}\rangle$ so $\nabla f(0,0)$ is undefined). This point goes with a relative (in fact, absolute) minimum (the vertex of this cone).

2. (9 points) Use a double Riemann sum to approximate $\iint\limits_R \sqrt{x+y^3}\,dA$ where $R=[-1,5]\times[-2,6]$. Use midpoint rule and a 3×2 grid of rectangles (3 across and 2 up) to partition R. (Don't worry about simplifying.)



We must split $-1\le x\le 5$ into 3 pieces. This interval has width $5-(-1)=6$, so $\Delta x=6/3=2$. We get partition points of $x_0=-1$, $x_1=1$, $x_2=3$, and $x_3=5$. This means our midpoints are located at $x=0,2,4$. Likewise, we must split $-2\le y\le 6$ into 2 pieces. This interval is $6-(-2)=8$ units long, so $\Delta y=8/2=4$. We get partition points of $y_0=-2$, $y_1=2$, and $y_2=6$. This means out midpoints are located at $y=0,4$. This our sample points (to be plugged into $\sqrt{x+y^3}$ are $(0,0),(0,4),(2,0),(2,4),(4,0)$, and $(4,4)$.

$$\text{integral} \approx 2\cdot 4\cdot\left(\sqrt{0+0^3}+\sqrt{0+4^3}+\sqrt{2+0^3}+\sqrt{2+4^3}+\sqrt{4+0^3}+\sqrt{4+4^3}\right)$$

3. (9 points) Let $z=f(x,y)$, $x=u+v$, and $y=u-v$. Show that $\left(\frac{\partial z}{\partial x}\right)^2-\left(\frac{\partial z}{\partial y}\right)^2=\frac{\partial z}{\partial u}\cdot\frac{\partial z}{\partial v}$. [Use the chain rule.]

$$\frac{\partial z}{\partial u}\cdot\frac{\partial z}{\partial v}=\left(\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}\right)\left(\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}\right)=\left(\frac{\partial z}{\partial x}\cdot(1)+\frac{\partial z}{\partial y}\cdot(1)\right)\left(\frac{\partial z}{\partial x}\cdot(1)+\frac{\partial z}{\partial y}\cdot(-1)\right)=\left(\frac{\partial z}{\partial x}\right)^2-\left(\frac{\partial z}{\partial y}\right)^2$$

4. (10 points) Limits.

(a) Show that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{3x^2+x^2y+3y^2}{x^2+y^2}$ does exist and find this limit.

We cannot just plug in $(x,y)=(0,0)$ since that will cause a division by zero. Algebra won't get rid of this problem, so our only remaining tool for showing limits exist is a change of coordinates. Let's change to polar coordinates.

$$\lim_{(x,y)\to(0,0)} \frac{3x^2+x^2y+3y^2}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \frac{3(x^2+y^2)+x^2y}{x^2+y^2} = \lim_{(r,\theta)\to(0,\theta)} \frac{3r^2+r^2\cos^2(\theta)\cdot r\sin(\theta)}{r^2}$$
$$= \lim_{(r,\theta)\to(0,\theta)} 3+r\cos^2(\theta)\sin(\theta) = 3+0=3$$

Note: When approaching the origin in polar coordinates, $r=0$ but θ is undetermined. Next, notice that since sine and cosine are bounded by ±1, $r\cos^2(\theta)\sin(\theta)$ is bounded by $\pm r$ so as $r\to 0$, $r\cos^2(\theta)\sin(\theta)$ is squeezed between $\pm r \to \pm 0 = 0$ (and so this term also goes to zero).

(b) Show that $\displaystyle\lim_{(x,y,z)\to(0,0,0)} \frac{xy+xz}{x^2+y^2+z^2}$ does not exist.

We shoot down limits by finding two curves that give us incompatible limits. Since this is a limit in 3-space, we must parameterize our curves. First, let's approach along the x-axis: $\mathbf{r}(t)=\langle t,0,0\rangle$. Our limit becomes $\displaystyle\lim_{t\to0}\frac{t(0)+t(0)}{t^2+0^2+0^2} = \lim_{t\to0}\frac{0}{t^2}=0$. Unfortunately, approaching along the other coordinate axes will give us the same answer. The next thing I usually try is approaching along a curve whose components "unify the denominator". Consider $\mathbf{r}(t)=\langle t,t,t\rangle$ (this goes through the origin when $t=0$). We get $\displaystyle\lim_{t\to0}\frac{t(t)+t(t)}{t^2+t^2+t^2} = \lim_{t\to0}\frac{2t^2}{3t^2}=\frac{2}{3}$. Since approaching along one curve gave us 0 and along another curve gave us 2/3, this limit cannot exist.

5. (11 points) Let $F(x,y,z)=x\sin(yz)+ze^{xy}$. *Note:* All three parts use the same function and point.

(a) Find an equation for the plane tangent to $x\sin(yz)+ze^{xy}=1$ at $(x,y,z)=(-2,0,1)$

Note: $F(-2,0,1)=-2\sin(0)+1e^0=1$ so $(-2,0,1)$ does actually lie on the level surface $F(x,y,z)=x\sin(yz)+ze^{xy}=1$.

Recall that the gradient is perpendicular to level things, so $\nabla F(x,y,z)$ will be perpendicular to the level surfaces of $F(x,y,z)$. In particular, $\nabla F(-2,0,1)$ is a normal vector for the plane tangent to $F(x,y,z)=x\sin(yz)+ze^{xy}=1$ at $(x,y,z)=(-2,0,1)$.

$$\nabla F(x,y,z)=\langle F_x,F_y,F_z\rangle = \langle \sin(yz)+yze^{xy}, xz\cos(yz)+xze^{xy}, xy\cos(yz)+e^{xy}\rangle$$
$$\nabla F(-2,0,1) = \langle \sin(0)+0e^0, (-2)1\cos(0)+(-2)1e^0, 0\cos(0)+e^0\rangle = \langle 0,-4,1\rangle$$
$$\text{Our tangent plane:} \quad 0(x+2)-4(y-0)+1(z-1)=0 \quad \Longrightarrow \quad -4y+z-1=0$$

(b) Find the directional derivative $D_{\mathbf{u}}F(-2,0,1)$ where $\mathbf{u}$ points in the same direction as $\mathbf{v}=\langle -2,1,6\rangle$.

$\mathbf{u}$ needs to be a unit vector: $\mathbf{u}=\dfrac{\mathbf{v}}{|\mathbf{v}|}=\dfrac{1}{\sqrt{(-2)^2+1^2+6^2}}\langle -2,1,6\rangle = \dfrac{1}{\sqrt{41}}\langle -2,1,6\rangle$. We also need $\nabla F(-2,01)$, but this was computed in part (a).

$$D_{\mathbf{u}}F(-2,0,1)=\nabla F(-2,0,1)\bullet\mathbf{u} = \langle 0,-4,1\rangle \bullet \frac{1}{\sqrt{41}}\langle -2,1,6\rangle = \frac{0(-2)+(-4)(1)+1(6)}{\sqrt{41}} = \frac{2}{\sqrt{41}}$$

(c) Could $D_{\mathbf{u}}F(-2,0,1)=-4$ for some direction $\mathbf{u}$? $\boxed{\textbf{Yes}}$ / **No** Why or why not?

Recall that the directional derivative at a point is maximized and minimized by plus or minus the magnitude of the gradient at that point (and in fact achieves every value between the min and max). Note that $|\nabla F(-2,0,1)|=|\langle 0,-4,1\rangle| = \sqrt{0^2+(-4)^2+1^2}=\sqrt{17}$. Thus $-\sqrt{17}\le D_{\mathbf{u}}F(-2,0,1)\le\sqrt{17}$. Certainly, $-\sqrt{17}<-\sqrt{16}=-4<\sqrt{17}$, so -4 is a possible value.

6. (7 points) Cute 'n Fluffy Corp. ships their stuffed animals in large boxes with square bases. When these boxes are manufactured the side of the square base can vary by 3% and the height of the box can vary by 2%. Use a total derivative to estimate the total possible variance in the volume of such boxes.

The volume of an $x \times x \times y$ square based box is $V = x^2y$. Then $dV = V_x dx + V_y dy = 2xy\,dx + x^2\,dy$ can be thought of as the difference between the target box's volume and the actual manufactured box's volume. Thus $\frac{dV}{V} = \frac{2xy\,dx + x^2\,dy}{x^2y} = \frac{2xy\,dx}{x^2y} + \frac{x^2\,dy}{x^2y} = 2\frac{dx}{x} + \frac{dy}{y}$ represents the percent error in volume. We have been told that the percent error in the base is at most 3%: $|dx/x| \leq 3\%$ and the percent error in height is at most 2%: $|dy/y| \leq 2\%$. Therefore,

$$\left|\frac{dV}{V}\right| \leq \left|2\frac{dx}{x} + \frac{dy}{y}\right| \leq 2\left|\frac{dx}{x}\right| + \left|\frac{dy}{y}\right| \leq 2(3\%) + 2\% = 8\%.$$

7. (7 points) Let $xy^2z^3 + e^{x+2y}\sin(xyz) = 10$. Assuming that z is a function of x and y, compute $\frac{\partial z}{\partial x}$.

Recall our implicit diff. formula: If $F(x,y,z)$ =constant, then $\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}$. Let $F(x,y,z) = xy^2z^3 + e^{x+2y}\sin(xyz) = 10$.

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{y^2z^3 + e^{x+2y}\sin(xyz) + yze^{x+2y}\cos(xyz)}{3xy^2z^2 + xye^{x+2y}\cos(xyz)}$$

8. (7 points) Suppose we have a function of two variables: $f(x,y)$.

(a) It is possible for $f_{xy}(-1,2) = 5$ and $f_{yx}(-1,2) = 9$? If not, why not? If so, what does this tell us?

Clairaut's theorem states that if the mixed partial derivatives f_{xy} and f_{yx} are continuous, then they must be equal: $f_{xy} = f_{yx}$. So, $\boxed{\text{yes}}$, it is possible for $f_{xy}(-1,2) = 5 \neq 9 = f_{yx}(-1,2)$, but only if f_{xy} and/or f_{yx} is *not continuous* at $(x,y) = (-1,2)$.

(b) Suppose that f_x and f_y exist everywhere. Can I conclude f is differentiable? **YES** / $\boxed{\textbf{NO}}$

Differentiability implies that the first partials exist, but the converse does not necessarily hold. There are functions whose first partials exists, yet they fail to be differentiable.

(c) Suppose that f is differentiable. Can I conclude that f is continuous? $\boxed{\textbf{YES}}$ / **NO**

This is one of our theorems.

9. (10 points) Let $f(x,y) = x^3 - xy + y + 1$.

(a) Compute the gradient and Hessian matrix for f.

$$\nabla f(x,y) = \langle f_x, f_y \rangle = \langle 3x^2 - y, -x + 1 \rangle$$

$$H_f(x,y) = \begin{bmatrix} f_{xx} & f_{yx} \\ f_{xy} & f_{yy} \end{bmatrix} = \begin{bmatrix} 6x & -1 \\ -1 & 0 \end{bmatrix}$$

(b) Find the quadratic approximation of f at $(x,y) = (2,-1)$.

$f(2,-1) = 2^3 - 2(-1) + (-1) + 1 = 10$ $\quad \nabla f(2,-1) = \langle 3(2^2) - (-1), -2 + 1 \rangle = \langle 13, -1 \rangle$ $\quad H_f(2,-1) = \begin{bmatrix} 12 & -1 \\ -1 & 0 \end{bmatrix}$

$$\begin{aligned} Q(x,y) &= 10 + \langle 13, -1 \rangle \bullet \langle x-2, y+1 \rangle + \frac{1}{2}\begin{bmatrix} x-2 & y+1 \end{bmatrix}\begin{bmatrix} 12 & -1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x-2 \\ y+1 \end{bmatrix} \\ &= 10 + 13(x-2) - (y+1) + \frac{12}{2}(x-2)^2 + \frac{-1}{2}(x-2)(y+1) + \frac{-1}{2}(x-2)(y+1) + \frac{0}{2}(y+1)^2 \end{aligned}$$

(c) Find and classify all of the critical points of f. [Use the "2$^{\text{nd}}$-derivative" test to determine if critical points are relative max's, min's or saddle points.]

Critical points occur when ∇f is zero or does not exist. In our case, the gradient exists everywhere, so we should solve $\nabla f = \mathbf{0}$: $3x^2 - y = 0$ and $-x + 1 = 0$. The second equation says that $x = 1$ and so $3(1)^2 - y = 0$ implies $y = 3$. The only critical point is $(x,y) = (1,3)$. Let's test this point.

$H_f(1,3) = \begin{bmatrix} 6 & -1 \\ -1 & 0 \end{bmatrix}$. Then $\det(H_f(1,3) = 6(0) - (-1)(-1) = -1 < 0$. $\boxed{(1,3) \text{ is a saddle point.}}$

10. (10 points) Suppose $f(x,y)$ is a "nice" function (with continuous partials of all orders).

(a) $Q(x,y) = 11 - 5(x+4)^2 + 2(x+4)(y-6) - 3(y-6)^2$ is the quadratic approx. at $(x,y) = (-4,6)$.

Recall that the quadratic approximation of f at (a,b) is $Q(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) + \frac{1}{2}f_{xx}(a,b)(x-a)^2 + \frac{1}{2}f_{xy}(x-a)(y-b) + \frac{1}{2}f_{yx}(x-a)(y-b) + \frac{1}{2}f_{yy}(a,b)(y-b)^2$. This means in the above approximation, we must have $f(-4,6) = 11$. There are no first order terms, so the first partials must be zero. The coefficient of $(x+4)^2$ is -5, so $\frac{1}{2}f_{xx}(-4,6) = -5$ and thus $f_{xx}(-4,6) = -10$. Likewise, $f_{yy}(-4,6) = -6$. The mixed partials are a little different. Clairaut's theorem tells us that they're equal (all partials are assumed to be continuous in this problem). Thus, $2(x+4)(y-6) = \frac{1}{2}(f_{xy}(-4,6) + f_{yx}(-4,6))(x+4)(y-6) = \frac{1}{2}(f_{xy}(-4,6) + f_{xy}(-4,6))(x+4)(y-6) = f_{xy}(-4,6)(x+4)(y-6)$, so $f_{xy}(-4,6) = f_{yx}(-4,6) = 2$.

$$\nabla f(-4,6) = \langle 0,0 \rangle \qquad H_f(-4,6) = \begin{bmatrix} -10 & 2 \\ 2 & -6 \end{bmatrix}$$

Is $(x,y) = (-4,6)$ a critical point of $f(x,y)$? $\boxed{\textbf{YES}}$ / **NO**

If not, why not? If so, what kind (relative min, relative max, saddle point or not enough information)?

We found that $\nabla f(-4,6) = \mathbf{0}$, so this is a critical point. Next, $\det(H_f(-4,6)) = -10(-6) - 2(2) = 60 - 4 = 56 > 0$ and $f_{xx}(-4,6) = -10 < 0$, so $\boxed{(-4,6) \text{ is a relative maximum.}}$

(b) $Q(x,y) = x - 5(y-3) + 8x^2 - 3x(y-3) - (y-3)^2$ is the quadratic approx. at $(x,y) = (0,3)$.

$$\nabla f(0,3) = \langle 1,-5 \rangle \qquad H_f(0,3) = \begin{bmatrix} 16 & -3 \\ -3 & -2 \end{bmatrix}$$

Is $(x,y) = (0,3)$ a critical point of $f(x,y)$? **YES** / $\boxed{\textbf{NO}}$

If not, why not? If so, what kind (relative min, relative max, saddle point or not enough information)?

Since $\nabla f(0,3) \neq \mathbf{0}$, $(0,3)$ is not a critical point.

11. (10 points) Use the method of Lagrange multipliers to find the minimum and maximum values of

$$f(x,y,z) = xyz \text{ constrained to } 4x^2 + 2y^2 + z^2 = 12.$$

Note: Our constraint $g(x,y,z) = 4x^2 + 2y^2 + z^2 = 12$ is an ellipsoid. Since this is a *compact* set (i.e. closed and bounded) and $f(x,y,z) = xyz$ is continuous, the extreme value theorem guarantees that a min and max actually exist.

The Lagrange multiplier equations are $\nabla f = \lambda \nabla g$ and $g(x,y,z) = 12$. So $\langle yz, xz, xy \rangle = \lambda \langle 8x, 4y, 2z \rangle$ and we get...

$$\begin{aligned} yz &= 8x\lambda \\ xz &= 4y\lambda \\ xy &= 2z\lambda \\ 4x^2 + 2y^2 + z^2 &= 12 \end{aligned}$$

The best way to solve these equations is symmetrization. Multiply the first equation by x, the second by y, and the third by z and get $xyz = 8x^2\lambda = 4y^2\lambda = 2z^2\lambda$.

Let's take care of a technical issue. Can we have $\lambda = 0$? If so, $\nabla f = 0\nabla g = \mathbf{0}$. This would imply that $yz = xz = xy = 0$ and so $f(x,y,z) = xyz = 0$. Clearly, $f(x,y,z)$ takes on both positive and negative values when constrained to the ellipsoid $g(x,y,z) = 12$, so any point which goes with $\lambda = 0$ is uninteresting for us (it's not a min or max). Let's now assume $\lambda \neq 0$.

We get $\dfrac{xyz}{2\lambda} = 4x^2 = 2y^2 = z^2$. Therefore, $12 = 4x^2 + 2y^2 + z^2 = 4x^2 + 4x^2 + 4x^2 = 12x^2$ so that $x^2 = 1$ and so $x = \pm 1$. Next, $12 = 4x^2 + 2y^2 + z^2 = 2y^2 + 2y^2 + 2y^2 = 6y^2$ so that $y^2 = 2$ and so $y = \pm\sqrt{2}$. Finally, $12 = 4x^2 + 2y^2 + z^2 = z^2 + z^2 + z^2 = 3z^2$ so that $z^2 = 4$ and so $z = \pm 2$.

Therefore, we have 8 solutions: $(x,y,z) = (\pm 1, \pm\sqrt{2}, \pm 2)$ (allowing all possible sign choices). Plugging these in, we get $f(\pm 1, \pm\sqrt{2}, \pm 2) = \pm 1 \cdot \sqrt{2} \cdot 2 = \boxed{\pm 2\sqrt{2}}$. Thus the maximum value of $f(x,y,z) = xyz$ constrained to $4x^2 + 2y^2 + z^2 = 12$ is $2\sqrt{2}$ and the minimum value is $-2\sqrt{2}$. One final note, the max occurs at 4 of these critical points and the min at the other 4 points.