Math 2130 **Test #3** **July 27th, 2016**

Name: ANSWER KEY **Be sure to show your work!**

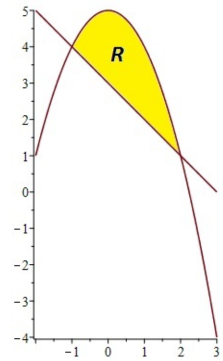
1. (12 points) Let R be the region bounded by $y = 5 - x^2$ and $y = -x + 3$.

[Warning: One of the integrals below will have to be split into 2 pieces.]

(a) Sketch the region R.

(b) Set up the integral $\iint_R f(x,y)\,dA$ using the order of integration "$dy\,dx$". [*Note:* You cannot evaluate these integrals.]

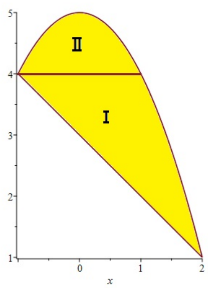
(c) Set up the integral $\iint_R f(x,y)\,dA$ using the order of integration "$dx\,dy$".



The region is bounded by a parabola opening downward and a line (as pictured to the right). We need to know the points of intersection, so we set the bounds equal to each other: $5 - x^2 = -x + 3$. Thus $x^2 - x - 2 = 0$ so $(x-2)(x+1) = 0$. This means the graphs cross when $x = -1$ and $x = 2$. Notice that $5 - (-1)^2 = 4 = -(-1) + 3$ and $5 - 2^2 = 1 = -2 + 3$. Thus the points of intersection are $(x, y) = (-1, 4)$ and $(2, 1)$.

First, as a y-simple region, the line is the bottom and the parabola is the top. So $-x+3 \leq y \leq 5 - x^2$. The points of intersect tell us that $-1 \leq x \leq 2$. Therefore,

$$\text{(b)} \quad \iint_R f(x,y)\,dA = \int_{-1}^{2} \int_{-x+3}^{5-x^2} f(x,y)\,dy\,dx$$



Treating this region as x-simple is a bit more difficult. Notice that the parabola is the right side of the whole region, but the left side is given by the line part of the time and the parabola part of the time. Let's split the region into two pieces (as pictured to the left).

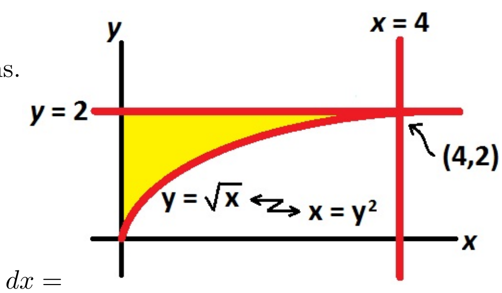
The right boundary is given by the parabola $y = 5 - x^2$. This needs to be an x-bound, so we solve for x: $x = \pm\sqrt{5-y}$. The positive branch is the right branch. For the left boundary, in part I it is given by the line $y = -x + 3$, so solving for x: $x = -y + 3$. For part II, the left bound comes from the parabola, so $x = -\sqrt{5-y}$.

We also need y-bounds. Notice that part I goes from the right point of intersection (we already found that $y = 1$ there) up to the left point of intersection (we found $y = 4$ there). The second region then goes from $y = 4$ up to the vertex of the parabola (which occurs when $y = 5$).

$$\text{(c)} \quad \iint_R f(x,y)\,dA = \int_1^4 \int_{-y+3}^{\sqrt{5-y}} f(x,y)\,dx\,dy + \int_4^5 \int_{-\sqrt{5-y}}^{\sqrt{5-y}} f(x,y)\,dx\,dy$$

2. (12 points) Evaluate $\displaystyle\int_0^4 \int_{\sqrt{x}}^{2} \sin(y^3)\,dy\,dx$.

Note/Hint: You cannot integrate $\int \sin(y^3)\,dy$ in terms of elementary functions.



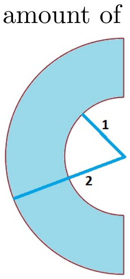
Reading the bounds of integration we get: $\sqrt{x} \leq y \leq 2$ and $0 \leq x \leq 4$ (notice the region is *above* the square root).

We need to reverse the order of integration so that we can evaluate the integral. As an x-simple region, we have $0 \leq x \leq y^2$ and $0 \leq y \leq 2$ (remember left-to-right and then bottom-to-top). Thus we get $\displaystyle\int_0^4 \int_{\sqrt{x}}^{2} \sin(y^3)\,dy\,dx =$

$$\int_0^2 \int_0^{y^2} \sin(y^3)\,dx\,dy = \int_0^2 x\sin(y^3)\Big|_0^{y^2} dy = \int_0^2 y^2 \sin(y^3)\,dy = -\frac{1}{3}\cos(y^3)\Big|_0^2 = -\frac{1}{3}\cos(2^3) + \frac{1}{3}\cos(0^3) = \frac{1-\cos(8)}{3}$$

3. (12 points) Let R be the left half (i.e. $x \leq 0$) of the annulus lying between $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

Sketch a picture of R then compute its centroid. You should be able to use symmetry and geometry to cut down the amount of integration you need to do.



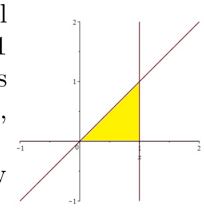
The area of this annulus is $m = \dfrac{\pi 2^2 - \pi 1^2}{2} = \dfrac{3}{2}\pi$. Symmetry tells us that $\bar{y} = 0$. So we really only need to compute one integral (M_y). This region is perfectly suited for polar coordinates. Imagine a ray emanating from the origin. It will enter R when it crosses the circle $x^2 + y^2 = 1$ (i.e. $r = 1$) and exit when it crosses $x^2 + y^2 = 4$ (i.e. $r = 2$). Therefore, $1 \leq r \leq 2$. To get only the right half of the annulus we should restrict θ: $\pi/2 \leq \theta \leq 3\pi/2$.

$$M_y = \iint_R x\,dA = \int_{\pi/2}^{3\pi/2} \int_1^2 r\cos(\theta)\cdot r\,dr\,d\theta = \int_{\pi/2}^{3\pi/2} \cos(\theta)\,d\theta \cdot \int_1^2 r^2\,dr = -2 \cdot \left[\frac{r^3}{3}\right|_1^2 = -\frac{2}{3}(8-1) = -\frac{14}{3}$$

Therefore, $\bar{x} = \dfrac{M_y}{m} = \dfrac{-14/3}{3\pi/2} = -\dfrac{28}{9\pi}$ so $\boxed{(\bar{x}, \bar{y}) = \left(-\dfrac{28}{9\pi}, 0\right)}$.

4. (13 points) Compute $\displaystyle\iint_R (x+y)e^{x-y}\,dA$ where R is the region bounded by $x+y=1$, $x-y=0$ and $y=0$.

Use a (natural) change of coordinates which simplifies the region R and… don't forget the Jacobian!



Considering the function being integrated and the bounds for R, $u = x+y$ and $v = x-y$ make a natural change of coordinates (this will simplify our integrand to ue^v). The bounds for R become $u = x+y = 1$, $v = x-y = 0$, and then $y = 0$ means $u = x+0 = x-0 = v$. So our new region (in the uv-plane) is bounded by $u = 1$, $v = 0$, and $v = u$. This new region is pictured to the right. It is both u- and v-simple, so we'll just treat it as v-simple and get bounds: $0 \le v \le u$ and $0 \le u \le 1$.
Next, we need to compute the Jacobian. Now our change of coordinates is "backwards" since we have new in terms of old coordinates. But this is easily dealt with…

$$\frac{\partial(u,v)}{\partial(x,y)} = \det\begin{bmatrix} u_x & v_x \\ u_y & v_y \end{bmatrix} = \det\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = 1(-1) - 1(1) = -2 \qquad \Longrightarrow \qquad J = \frac{\partial(x,y)}{\partial(u,v)} = 1 \Big/ \frac{\partial(u,v)}{\partial(x,y)} = -\frac{1}{2}$$

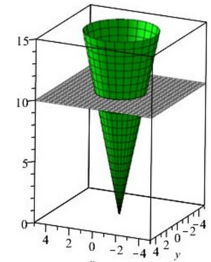
Alternatively, we could have solved for x and y. Adding $u = x+y$, $v = x-y$ together yields $u+v = 2x$ so that $x = \frac{1}{2}u + \frac{1}{2}v$. Subtracting yields $u - v = 2y$ so that $y = \frac{1}{2}u - \frac{1}{2}v$. From this we could compute the Jacobian directly (again we will get $J = -1/2$).

$$\iint_R (x+y)e^{x-y}\,dA = \int_0^1 \int_0^u ue^v \left|-\frac{1}{2}\right| dv\,du = \int_0^1 \frac{u}{2}e^v\Big|_0^u\,du = \int_0^1 \frac{1}{2}ue^u - \frac{1}{2}ue^0\,du$$

Note: To integrate $\int ue^u\,du$ we use integration by parts. In detail, in $\int te^t\,dt$ use $u = t$ and $dv = e^t\,dt$ then $du = dt$ and $v = e^t$. Thus $\int te^t\,dt = \int u\,dv = uv - \int v\,du = te^t - \int e^t\,dt = (t-1)e^t + C$. Returning to our calculation…

$$= \frac{1}{2}(u-1)e^u - \frac{1}{4}u^2\Big|_0^1 = \left(\frac{1}{2}(1-1)e^1 - \frac{1^2}{4}\right) - \left(\frac{1}{2}(0-1)e^0 - \frac{0^2}{4}\right) = -\frac{1}{4} + \frac{1}{2} = \frac{1}{4}$$

5. (15 points) Let E be the region bounded by $z = 5\sqrt{x^2+y^2}$ and $z = 10$. Set up integrals which computes the volume of E using the following orders of integration: [Do **not** evaluate these integrals.]



(a) $\int_?^? \int_?^? \int_?^? ???\,dz\,dy\,dx$

(b) $\int_?^? \int_?^? \int_?^? ???\,dx\,dz\,dy$

(c) Set up this integral in cylindrical coordinates.

(d) Set up this integral in spherical coordinates.

Note that $\displaystyle\iiint_E 1\,dV$ computes the volume. A sketch of this region should be helpful. Notice that $z = 5\sqrt{x^2+y^2}$ is a cone (level curves are circles and the vertical trace $y = 0$ yields $z = 5\sqrt{x^2+0^2} = 5|x|$). This cone is topped off at $z = 10$.

First, treating this as a z-simple region we have $5\sqrt{x^2+y^2} \le z \le 10$ (cone up to plane). Then projecting out the z-coordinate yields a disk in the xy-plane whose boundary is determined by where the plane and cone intersect. This is $10 = 5\sqrt{x^2+y^2}$ so $\sqrt{x^+y^2} = 2$ which is $x^2+y^2 = 4$ (a circle of radius 2). Next, we need y-bounds. Solving for y gets us $y = \pm 4 - x^2$ (a lower and upper semicircle). Finally, $-2 \le x \le 2$ (since the radius is 2).

$$\text{(a)} \qquad \iiint_E 1\,dV = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{5\sqrt{x^2+y^2}}^{10} 1\,dz\,dy\,dx$$

Next, treating this as a x-simple region, we can see that the cone determines the back and front of E. So solving $z = 5\sqrt{x^2+y^2}$ for x yields out x-bounds: $-\sqrt{-y^2+z^2/25} \le x \le \sqrt{-y^2+z^2/25}$. Projecting out the x-coordinate, we get a triangular shaped region in the yz-plane. The top is determined by the plane: $z = 10$. The bottom is determined by where front and back collide. This is the same as plugging $x = 0$ in the cone's equation: $z = 5\sqrt{0^2+y^2} = 5|y|$. Finally, the y-bounds come from where the z-bounds intersect: $10 = 5|y|$ so $|y| = 2$ and thus $y = \pm 2$.

$$\text{(b)} \qquad \iiint_E 1\,dV = \int_{-2}^{2} \int_{5|y|}^{10} \int_{-\sqrt{-y^2+z^2/25}}^{\sqrt{-y^2+z^2/25}} 1\,dx\,dz\,dy$$

For cylindrical coordinates we look back at part (a). We have $5r = 5\sqrt{x^2+y^2} \le z \le 10$. In the xy-plane we had a disk of radius 2, so $0 \le r \le 2$ and $0 \le \theta \le 2\pi$. Don't forget the Jacobian.

$$\text{(c)} \quad \iiint\limits_E 1\,dV = \int_0^{2\pi}\int_0^2\int_{5r}^{10} r\,dz\,dr\,d\theta$$

Finally, for spherical coordinates, imagine a ray emanating out of the origin. We start off inside E so the lower bound for ρ is 0. The ray will exit E as it crosses through the plane $z = 10$. Converting to spherical coordinates, we have $\rho\cos(\phi) = 10$ so $\rho = 10/\cos(\phi) = 10\sec(\phi)$ is our upper bound. Our θ bound is the same as in cylindrical coordinates. Finally, imagine a ray sweeping down from the z-axis. We start off inside of E so the lower bound for ϕ is 0. We should sweep down until we hit the cone. So $z = 5\sqrt{x^2+y^2} = 5r$ determines the upper bound for ϕ. Recall that $r = \rho\sin(\phi)$, so $\rho\cos(\phi) = 5\rho\sin(\phi)$ and so $\frac{1}{5} = \frac{\sin(\phi)}{\cos(\phi)} = \tan(\phi)$. Therefore, $\phi = \arctan(\frac{1}{5})$. Alternatively, we could have found the cone's angle by drawing a triangular cross-section of the region E.

$$\text{(d)} \quad \iiint\limits_E 1\,dV = \int_0^{2\pi}\int_0^{\arctan(1/5)}\int_0^{10\sec(\phi)} \rho^2\sin(\phi)\,d\rho\,d\phi\,d\theta$$

We were not asked to actually compute the volume of E, but if wanted to do this, cylindrical coordinates would be best.

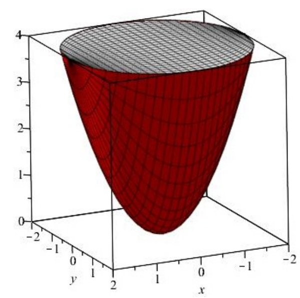
6. (12 points) Compute $\iiint_E \dfrac{1}{\sqrt{x^2+y^2+z^2}}\,dV$ where E is the solid region outside $x^2+y^2+z^2=1$, inside $x^2+y^2+z^2=4$, and in the first octant.

We should use spherical coordinates – we are integrating a region trapped between spheres and our integrand is $1/\sqrt{x^2+y^2+z^2} = 1/\rho$. We have ρ bounds $\rho^2 = x^2+y^2+z^2 = 1$ and $\rho^2 = x^2+y^2+z^2 = 4$, so $1 \le \rho \le 2$. Finally, we are restricted to the first octant, so $0 \le \phi \le \pi/2$ (upper-half of 3-space) and $0 \le \theta \le \pi/2$ (first quadrant in the polar plane).

$$\iiint\limits_E \frac{1}{\sqrt{x^2+y^2+z^2}}\,dV = \int_0^{\pi/2}\int_0^{\pi/2}\int_1^2 \frac{1}{\rho}\cdot\rho^2\sin(\phi)\,d\rho\,d\phi\,d\theta = \int_0^{\pi/2} d\theta\cdot\int_0^{\pi/2}\sin(\phi)\,d\phi\cdot\int_1^2 \rho\,d\rho = \frac{\pi}{2}\cdot 1\cdot\frac{2^2-1^2}{2} = \frac{3\pi}{4}$$

7. (12 points) Consider the region E bounded below by $z = x^2+y^2$ and above by $z = 4$. Compute the centroid of E.

Free information: The volume of E is 8π.



We have been given $m = 8\pi$ and can see that by symmetry $\bar{x} = \bar{y} = 0$. This only leaves M_{xy} to be computed. Given the symmetry of the region, cylindrical coordinates should be a good choice. Notice that $r^2 = x^2+y^2 \le z \le 4$ (z is bounded below by the paraboloid and above by the plane). If we project out the z-coordinate, we get a disk in the xy-plane. This disk is determined by the circle coming from where the paraboloid and plane intersect: $r^2 = 4$ so $r = 2$. Thus $0 \le r \le 2$ and $0 \le \theta \le 2\pi$.

$$M_{xy} = \iiint\limits_E z\,dV = \int_0^{2\pi}\int_0^2\int_{r^2}^4 z\cdot r\,dz\,dr\,d\theta = \int_0^{2\pi} d\theta\cdot\int_0^2\int_{r^2}^4 rz\,dz\,dr = 2\pi\cdot\int_0^2 \frac{1}{2}rz^2\Big|_{r^2}^4\,dr$$

$$= 2\pi\cdot\int_0^2 8r - \frac{1}{2}r^5\,dr = 2\pi\left[4r^2 - \frac{1}{12}r^6\right]_0^2 = 2\pi\left(16 - \frac{16}{3}\right) = \frac{64}{3}\pi. \text{ Therefore, } \bar{z} = \frac{M_{xy}}{m} = \frac{64\pi/3}{8\pi} = \frac{8}{3}.$$

The centroid of E is $(\bar{x},\bar{y},\bar{z}) = \left(0, 0, \dfrac{8}{3}\right)$.

8. (12 points) Consider the integral: $I = \displaystyle\int_0^7\int_{-\sqrt{49-x^2}}^0\int_{-\sqrt{49-x^2-y^2}}^0 z\cos(x^2+y^2+z^2)\,dz\,dy\,dx.$

This is part of a ball of radius 7. The z-bounds show we are dealing with part of the lower-half (i.e. $\pi/2 \le \phi \le \pi$). The x- and y-bounds describe the part of a disk in the fourth quadrant (i.e. $3\pi/2 \le \theta \le 2\pi$).

(a) Rewrite I in the following order of integration…

$$I = \int_0^7\int_{-\sqrt{49-x^2}}^0\int_{-\sqrt{49-x^2-z^2}}^0 z\cos(x^2+y^2+z^2)\,dy\,dz\,dx$$

(b) Rewrite I in terms of cylindrical coordinates.

$$I = \int_{3\pi/2}^{2\pi}\int_0^7\int_{-\sqrt{49-r^2}}^0 z\cos(r^2+z^2)\cdot r\,dz\,dr\,d\theta$$

(c) Rewrite I in terms of spherical coordinates.

$$I = \int_{3\pi/2}^{2\pi}\int_{\pi/2}^{\pi}\int_0^7 \rho\cos(\phi)\cos(\rho^2)\cdot\rho^2\sin(\phi)\,d\rho\,d\phi\,d\theta$$