Math 2130 **Final Exam** **June 28^{th}, 2019**

Name: <u>Answer Key</u> **Be sure to show your work!**

1. (7 points) Let $\mathbf{F}(x,y,z) = \langle x^2+yz, y+e^z, \sin(xy^2)\rangle$. Compute $\nabla\times\mathbf{F}$ and $\nabla\bullet\mathbf{F}$.

If $\mathbf{F}$ conservative? Yes / $\boxed{\text{No}}$

First, the curl of $\mathbf{F}$ is $\nabla\times\mathbf{F} =$

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2+yz & y+e^z & \sin(xy^2)\end{vmatrix} = \left\langle \frac{\partial}{\partial y}\left[\sin(xy^2)\right] - \frac{\partial}{\partial z}\left[y+e^z\right], -\left(\frac{\partial}{\partial x}\left[\sin(xy^2)\right] - \frac{\partial}{\partial z}\left[x^2+yz\right]\right), \frac{\partial}{\partial x}\left[y+e^z\right] - \frac{\partial}{\partial y}\left[x^2+yz\right]\right\rangle =$$

$\boxed{\nabla\times\mathbf{F} = \left\langle 2xy\cos(xy^2) - e^z, -y^2\cos(xy^2)+y, -z\right\rangle}$ ($\nabla\times\mathbf{F}\neq\mathbf{0}$ so $\mathbf{F}$ is not conservative.)

The Divergence of $\mathbf{F}$ is $\nabla\bullet\mathbf{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right\rangle \bullet \langle x^2+yz, y+e^z, \sin(xy^2)\rangle = \frac{\partial}{\partial x}\left[x^2+yz\right] + \frac{\partial}{\partial y}\left[y+e^z\right] + \frac{\partial}{\partial z}\left[\sin(xy^2)\right] =$

$\boxed{\nabla\bullet\mathbf{F} = 2x+1}$

2. (11 points) Let $\mathbf{F}(x,y,z) = \langle 2xyz+1, x^2z+z, x^2y+y+2z\rangle$. This is a conservative vector field (I've checked for you).

(a) Use the fundamental theorem of line integrals to compute $\int_C \mathbf{F}\bullet d\mathbf{r}$ where C is the line segment from $(1,-1,0)$ to $(1,2,2)$.

First, we construct a potential function: $f(x,y,z) = \int (2xyz+1)\,dx = x^2yz+x+C_1(y,z)$ and $f(x,y,z) = \int (x^2z+z)\,dy = x^2yz+yz+C_2(x,z)$ and $f(x,y,z) = \int (x^2y+y+2z)\,dz = x^2yz+yz+z^2+C_3(x,y)$. Thus $\underline{f(x,y,z) = x^2yz+x+yz+z^2}$ (plus any constant).

By the fundamental theorem of line integrals,

$$\int_C \mathbf{F}\bullet d\mathbf{r} = \int_C \nabla f\bullet d\mathbf{r} = f(1,2,2) - f(1,-1,0) = (4+1+4+4) - (0+1+0+0) = 13-1 = \boxed{12}$$

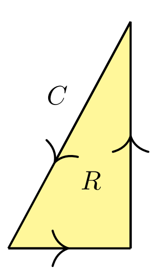
(b) Recompute $\int_C \mathbf{F}\bullet d\mathbf{r}$ directly (i.e. parameterize C etc.).

The standard parameterization of a line segment from A to B is $\mathbf{r}(t) = A+(B-A)t$ for $0\leq t\leq 1$. Thus we have $\mathbf{r}(t) = \langle 1,-1,0\rangle + \langle 0,3,2\rangle t = \langle 1, 3t-1, 2t\rangle$ for $0\leq t\leq 1$. The derivative of this parameterization is $\mathbf{r}'(t) = \langle 0,3,2\rangle$.

$$\int_C \mathbf{F}\bullet d\mathbf{r} = \int_0^1 \mathbf{F}(\mathbf{r}(t))\bullet d\mathbf{r} = \int_0^1 \langle 2(1)(3t-1)(2t)+1, 1^2(2t)+2t, 1^2(3t-1)+(3t-1)+2(2t)\rangle \bullet \langle 0,3,2\rangle\,dt$$

$$= \int_0^1 \langle 12t^2-4t+1, 4t, 10t-2\rangle\bullet\langle 0,3,2\rangle\,dt = \int_0^1 0+12t+2(10t-2)\,dt = \int_0^1 32t-4\,dt = 16t^2-4t\Big|_0^1 = 16-4 = \boxed{12}$$

3. (10 points) Let C be the boundary of a triangle with vertices $(0,-1)$, $(2,-1)$, and $(2,3)$ oriented counter-clockwise. Compute $\displaystyle\int_C \left(\cos(x^6+e^{3x})+y\right)\,dx + \left(x^2+\ln(y^8+3)\right)\,dy$.



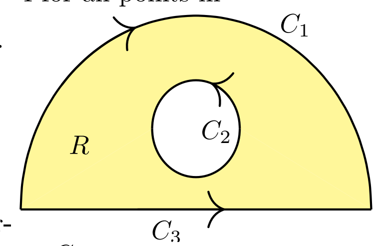
Notice that $C=\partial R$ (it's the properly oriented boundary of R). Thus we can apply Green's theorem (instead of working through 3 nasty line integrals along line segments).

This triangular region R is bounded below by the line $y=-1$, above by $y=2x-1$, and on the right by $x=2$. Thus we can describe it as a y-simple region as follows: $-1\leq y\leq 2x-1$ and $0\leq x\leq 2$. Then noting that $M(x,y) = \cos(x^6+e^{3x})+y$ and $N(x,y) = x^2+\ln(y^8+3)$, we have

$$\int_C M\,dx+N\,dy = \iint_R (N_x - M_y)\,dA = \int_0^2\int_{-1}^{2x-1} (2x-1)\,dy\,dx = \int_0^2 (2x-1)y\Big|_{-1}^{2x-1}\,dx$$

$$= \int_0^2 (2x-1)(2x-1)-(2x-1)(-1)\,dx = \int_0^2 4x^2-2x\,dx = \frac{4}{3}x^3-x^2\Big|_0^2 = \frac{32}{3}-4 = \boxed{\frac{20}{3}}$$

4. (9 points) C_1 is an upper-half of a circle of radius 3 (oriented clockwise), C_2 is a circle of radius 1 (oriented counter clockwise), and C_3 is a line segment closing off the semi-circle (oriented left to right). Let $\mathbf{F}(x,y) = \langle M(x,y), N(x,y)\rangle$ be a vector field such that M and N have continuous first partials and in addition, $N_x - M_y = 4$ for all points in region R. We also know $\int_{C_2} \mathbf{F} \bullet d\mathbf{r} = 2\pi$ and $\int_{C_3} \mathbf{F} \bullet d\mathbf{r} = \pi$. *Note:* The area of R is $7\pi/2$.



First, we could have figured out the area ourselves: a half disk of radius 3 has area $\dfrac{\pi 3^2}{2}$ and we remove the area of a disk of radius 1 which is $\pi 1^2$. Thus area is $\dfrac{9\pi}{2} - \pi = \dfrac{7\pi}{2}$.

Notice that $\partial R = C_3 - C_1 - C_2$ since we go around the outer boundary of R in a counter-clockwise direction: $C_3 - C_1$ and we go around an inner boundary in a clockwise direction: $-C_2$.

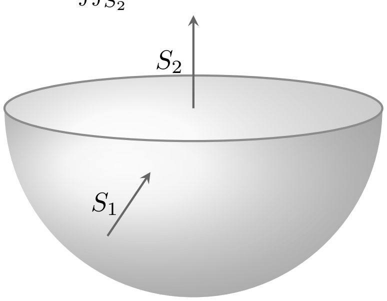
Green's (with holes) tells us that... (Don't forget $N_x - M_y = 4$.)

$$4 \cdot \text{Area}(R) = \iint_R 4\, dA = \iint_R (N_x - M_y)\, dA = \int_{\partial R} M dx + N dy = \int_{C_3} M dx + N dy - \int_{C_1} M dx + N dy - \int_{C_2} M dx + N dy$$

So $4 \cdot \frac{7\pi}{2} = \pi - \int_{C_1} M dx + N dy - 2\pi$. Therefore,

Compute $\displaystyle\int_{C_1} M(x,y)\, dx + N(x,y)\, dy \quad = \quad \underline{\pi - 2\pi - 14\pi = \boxed{-15\pi}}$.

5. (12 points) Let S_1 be the lower hemisphere: $x^2 + y^2 + z^2 = 1$, $z \leq 0$ and S_2 be the disk $x^2 + y^2 \leq 1$ in the xy-plane. Orient both S_1 and S_2 upward. Let $\mathbf{F}$ be a smooth vector field such that $\iint_{S_1} \mathbf{F} \bullet \mathbf{n}\, d\sigma = 10\pi$ and $\nabla \bullet \mathbf{F} = 5x^2 + 5y^2 + 5z^2$. Find $\iint_{S_2} \mathbf{F} \bullet \mathbf{n}\, d\sigma$.



We are going to "swap surfaces". Notice that $\partial E = S_2 - S_1$ (we need to orient outward) is the boundary of the solid lower ball E: $x^2 + y^2 + z^2 \leq 1$, $z \leq 0$. Thus we can apply the divergence theorem:

$$\iiint_E \nabla \bullet \mathbf{F}\, dV = \iint_{\partial E} \mathbf{F} \bullet \mathbf{n}\, d\sigma = \iint_{S_2} \mathbf{F} \bullet \mathbf{n}\, d\sigma - \iint_{S_1} \mathbf{F} \bullet \mathbf{n}\, d\sigma$$

$$\iint_{S_2} \mathbf{F} \bullet \mathbf{n}\, d\sigma = \iint_{S_1} \mathbf{F} \bullet \mathbf{n}\, d\sigma + \iiint_E \nabla \bullet \mathbf{F}\, dV = 10\pi + \iiint_E \nabla \bullet \mathbf{F}\, dV$$

since we were given $\iint_{S_1} \mathbf{F} \bullet \mathbf{n}\, d\sigma = 10\pi$. Thus we just need to compute the triple integral $\iiint_E \nabla \bullet \mathbf{F}\, dV = \iiint_E 5x^2 + 5y^2 + 5z^2\, dV$ to find our answer.

Obviously we should compute this integral using spherical coordinates. Then E: $\rho^2 = x^2 + y^2 + z^2 \leq 1$ gives $0 \leq \rho \leq 1$ and $z \leq 0$ (lower-half of 3-space) says $\pi/2 \leq \varphi \leq \pi$ and there's no need to cut down θ so $0 \leq \theta \leq 2\pi$. Finally, $5x^2 + 5y^2 + 5z^2 = 5\rho^2$ and don't forget the Jacobian!

$$\iiint_E 5x^2 + 5y^2 + 5z^2\, dV = \int_0^{2\pi} \int_{\pi/2}^{\pi} \int_0^1 5\rho^2 \cdot \rho^2 \sin(\varphi)\, d\rho\, d\varphi\, d\theta = \int_0^{2\pi} d\theta \int_{\pi/2}^{\pi} \sin(\varphi)\, d\varphi \int_0^1 5\rho^4\, d\rho = 2\pi \cdot 1 \cdot 1 = 2\pi$$

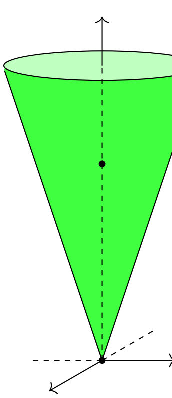
Therefore, $\displaystyle\iint_{S_2} \mathbf{F} \bullet \mathbf{n}\, d\sigma = \iint_{S_1} \mathbf{F} \bullet \mathbf{n}\, d\sigma + \iiint_E \nabla \bullet \mathbf{F}\, dV = 10\pi + 2\pi = \boxed{12\pi}$.

6. (12 points) Find the centroid of the part of the cone $z = 3\sqrt{x^2 + y^2}$ where $z \leq 6$.

Note: This is a **surface**. You should be computing **surface integrals**.

$$m = \iint_{S_1} 1\, d\sigma \qquad M_{yz} = \iint_{S_1} x\, d\sigma \qquad M_{xz} = \iint_{S_1} y\, d\sigma \qquad M_{xy} = \iint_{S_1} z\, d\sigma$$

Clearly, by symmetry $\bar{x} = \bar{y} = 0$ (the centroid lies on the z-axis). We need to compute $\bar{z} = M_{xy}/m$. Our first, step is to parameterize the cone. Our best choice is to use cylindrical coordinates. Then $z = 3\sqrt{x^2 + y^2} = 3r$, so $\mathbf{r}(r, \theta) = \langle r\cos(\theta), r\sin(\theta), 3r\rangle$. There's no need to cut down θ: $0 \leq \theta \leq 2\pi$. And we have $3r = z \leq 6$ so $0 \leq r \leq 2$.



Next, we need to compute our surface area element. Note that $\mathbf{r}_r = \langle \cos(\theta), \sin(\theta), 3\rangle$, $\mathbf{r}_\theta = \langle -r\sin(\theta), r\cos(\theta), 0\rangle$, and so $\mathbf{r}_r \times \mathbf{r}_\theta = \langle -3r\cos(\theta), -3r\sin(\theta), r\rangle = r\langle -3\cos(\theta), -3\sin(\theta), 1\rangle$. Thus $|\mathbf{r}_r \times \mathbf{r}_\theta| = r\sqrt{9\cos^2(\theta) + 9\sin^2(\theta) + 1} = r\sqrt{10}$.

$$m = \iint_{S_1} 1\, d\sigma = \int_0^{2\pi} \int_0^2 r\sqrt{10}\, dr\, d\theta = \sqrt{10} \int_0^{2\pi} d\theta \int_0^2 r\, dr = \sqrt{10} \cdot 2\pi \cdot \frac{2^2}{2} = 4\pi\sqrt{10}$$

$$M_{xy} = \iint_{S_1} z\, d\sigma = \int_0^{2\pi} \int_0^2 3r \cdot r\sqrt{10}\, dr\, d\theta = \sqrt{10} \int_0^{2\pi} d\theta \int_0^2 3r^2\, dr = \sqrt{10} \cdot 2\pi \cdot 2^3 = 16\pi\sqrt{10}$$

Therefore, $\bar{z} = M_{xy}/m = 16\pi\sqrt{10}/4\pi\sqrt{10} = 4$, so $\boxed{(\bar{x}, \bar{y}, \bar{z}) = (0, 0, 4)}$.

7. (13 points) Let S_1 be the surface parameterized by $\mathbf{r}(u,v) = \langle u\cos(v), u\sin(v), 9-u^2\rangle$ where $1 \le u \le 2$ and $\pi \le v \le 2\pi$.

Note: While we don't need to know this to complete this problem, it is interesting to note that S_1 is a part of the paraboloid $z = 9 - x^2 - y^2$ parameterized with cylindrical like coordinates.

(a) Find both orientations for S_1.

$\mathbf{r}_u = \langle \cos(v), \sin(v), -2u\rangle$, $\mathbf{r}_v = \langle -u\sin(v), u\cos(v), 0\rangle$, and so $\mathbf{r}_u \times \mathbf{r}_v = \langle 2u^2\cos(v), 2u^2\sin(v), u\rangle = u\langle 2u\cos(v), 2u\sin(v), 1\rangle$. Thus $|\mathbf{r}_u \times \mathbf{r}_v| = u\sqrt{4u^2\cos^2(v) + 4u^2\sin^2(v) + 1} = u\sqrt{4u^2+1}$.

Therefore, $\mathbf{n} = \pm\dfrac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|} = \boxed{\pm\dfrac{1}{\sqrt{4u^2+1}}\langle 2u\cos(v), 2u\sin(v), 1\rangle}$.

(b) Set up but **do not evaluate** the surface integral $\displaystyle\iint_{S_1} z\ln(x^2+y^2+1)\,d\sigma$. [Don't worry about simplifying.]

We already know that $d\sigma = |\mathbf{r}_u \times \mathbf{r}_v|\,dA = u\sqrt{4u^2+1}\,dA$ and notice that $x^2+y^2 = (u\cos(v))^2 + (u\sin(v))^2 = u^2$. Thus

$$\iint_{S_1} z\ln(x^2+y^2+1)\,d\sigma = \boxed{\int_\pi^{2\pi}\int_1^2 (9-u^2)\ln(u^2+1)\cdot u\sqrt{4u^2+1}\,du\,dv}.$$

(c) Set up but **do not evaluate** the flux integral $\displaystyle\iint_{S_1} \mathbf{F}\bullet\mathbf{n}\,d\sigma$ where S_1 is <u>oriented downward</u> and $\mathbf{F}(x,y,z) = \langle xz, y, x+y\rangle$. [Don't worry about computing the dot product or any significant simplification.]

We know that $\mathbf{n}\,d\sigma = \pm\dfrac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}|\mathbf{r}_u \times \mathbf{r}_v| = \pm\mathbf{r}_u \times \mathbf{r}_v$. Pointing upward or downward depends on the **k**-component of our orientation. Notice that this component is $\pm 1/\sqrt{4u^2+1}$ so we need to choose the minus sign to make the **k**-component negative and thus orient downward. Therefore,

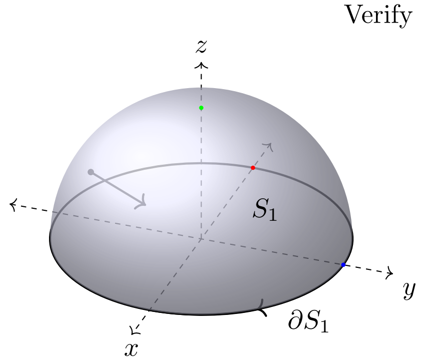
$$\boxed{\iint_{S_1} \mathbf{F}\bullet\mathbf{n}\,d\sigma = \int_\pi^{2\pi}\int_1^2 \langle u\cos(v)\cdot(9-u^2), u\sin(v), u\cos(v)+u\sin(v)\rangle \bullet \langle -2u^2\cos(v), -2u^2\sin(v), -u\rangle\,du\,dv}$$

8. (11 points) Let S_1 be the surface of the solid cylinder defined by $x^2+y^2 \le 4$ and $-2 \le z \le 3$. Orient S_1 outward and let $\mathbf{F}(x,y,z) = \left\langle 4xy^2 + \cos(y^{15}+z), \ln(x^2+z^4+1) + 2y, 4x^2z + e^{x^3y}\right\rangle$. Compute the flux integral $\iint_{S_1} \mathbf{F}\bullet\mathbf{n}\,d\sigma$.

Our surface is the surface of a solid (i.e., it's a closed surface) and oriented outward. This means we can apply the divergence theorem. Let E be this solid region: $x^2+y^2 \le 4$ and $-2 \le z \le 3$ so that $\partial E = S_1$. Also, notice that $\nabla\bullet\mathbf{F} = 4y^2 + 2 + 4x^2$. Considering our solid region and the intended integrand, cylindrical coordinates are an obvious choice. In cylindrical coordinates E is described by $r^2 \le 4$ (so $0 \le r \le 2$, $-2 \le z \le 3$, and $0 \le \theta \le 2\pi$. (Don't forget the Jacobian!)

$$\iint_{S_1} \mathbf{F}\bullet\mathbf{n}\,d\sigma = \iiint_E \nabla\bullet\mathbf{F}\,dV = \iiint_E 4(x^2+y^2) + 2\,dV = \int_0^{2\pi}\int_0^2\int_{-2}^3 (4r^2+2)\,r\,dz\,dr\,d\theta$$
$$= \int_0^{2\pi} d\theta \int_0^2 (4r^3+2r)\,dr \int_{-2}^3 dz = 2\pi\cdot(2^4+2^2)\cdot 5 = \boxed{200\pi}$$

9. (15 points) Let S_1 be the upper hemisphere: $x^2+y^2+z^2 = 4$ and $z \ge 0$. Orient S_1 downward. Verify Stokes' Theorem for the surface S_1, its boundary, and the vector field $\mathbf{F}(x,y,z) = \langle z, x, x\rangle$.



We've drawn the upper-hemisphere to the left noting the downward orientation. The boundary of S_1 is the equator $x^2+y^2+0^2 = 4$ with $z = 0$. The equator's induced orientation flows from the positive y-axis to the x-axis. This means that our circular boundary is <u>**oriented clockwise**</u>. [If it's hard to see why the arrow is drawn in that direction, consider the opposite orientation and think of someone walking on top of the sphere. If they keep the sphere to their left as they walk along the edge, they'll go in the counter-clockwise direction. Thus ours should be opposite that.]

We need to show: $\underbrace{\iint_{S_1} (\nabla\times\mathbf{F})\bullet\mathbf{n}\,d\sigma}_{\boxed{\text{I}}} = \underbrace{\int_{\partial S_1} \mathbf{F}\bullet d\mathbf{r}}_{\boxed{\text{II}}}$

I. First, $\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & x \end{vmatrix} = \langle 0, 0, 1 \rangle$. Next, we need to parameterize S_1 (use spherical coordinates). S_1 is part of the sphere $\rho^2 = x^2 + y^2 + z^2 = 4$ so $\rho = 2$ thus we parameterize with $\mathbf{r}(\varphi, \theta) = \langle 2\cos(\theta)\sin(\varphi), 2\sin(\theta)\sin(\varphi), 2\cos(\varphi) \rangle$ where $0 \leq \varphi \leq \pi/2$ (only the upper-hemisphere) and $0 \leq \theta \leq 2\pi$. Next, we need the "derivative" or our parameterization.

$\mathbf{r}_\varphi = \langle 2\cos(\theta)\cos(\varphi), 2\sin(\theta)\cos(\varphi), -2\sin(\varphi)$, $\mathbf{r}_\theta = \langle -2\sin(\theta)\sin(\varphi), 2\cos(\theta)\sin(\varphi), 0 \rangle$ and so $\mathbf{r}_\varphi \times \mathbf{r}_\theta = \langle 4\cos(\theta)\sin^2(\varphi), 4\sin(\theta)\sin^2(\varphi), 4\sin(\varphi)\cos(\varphi) \rangle$. Recall that $\mathbf{n}\, d\sigma = \pm \mathbf{r}_\varphi \times \mathbf{r}_\theta \, dA$. We wish to orient downward so once again we look at the $\mathbf{k}$-component: $4\sin(\varphi)\cos(\varphi)$. Now both $\sin(\varphi)$ and $\cos(\varphi)$ are non-negative when $0 \leq \varphi \leq \pi/2$, so we need to flip that sign to make the $\mathbf{k}$-component negative (i.e., point downward). Thus we should use $-\mathbf{r}_\varphi \times \mathbf{r}_\theta = \langle -4\cos(\theta)\sin^2(\varphi), -4\sin(\theta)\sin^2(\varphi), -4\sin(\varphi)\cos(\varphi) \rangle$ in our calculation. Now we're ready to write down our integral.

$$\iint_{S_1} (\nabla \times \mathbf{F}) \bullet \mathbf{n}\, d\sigma = \int_0^{2\pi} \int_0^{\pi/2} \langle 0, 0, 1 \rangle \bullet \langle -4\cos(\theta)\sin^2(\varphi), -4\sin(\theta)\sin^2(\varphi), -4\sin(\varphi)\cos(\varphi) \rangle \, d\varphi \, d\theta$$
$$= \int_0^{2\pi} \int_0^{\pi/2} -4\sin(\varphi)\cos(\varphi)\, d\varphi\, d\theta = \int_0^{2\pi} d\theta \int_0^{\pi/2} -4\sin(\varphi)\cos(\varphi)\, d\varphi = 2\pi \left[-2\sin^2(\varphi) \right|_0^{\pi/2} = \boxed{-4\pi}.$$

II. To compute the line integral side, we need to begin by parameterizing ∂S_1. Now S_1 is the circle $x^2 + y^2 = 4$, $z = 0$ oriented clockwise. Consider $\mathbf{r}(t) = \langle 2\cos(t), 2\sin(t), 0 \rangle$ with $0 \leq t \leq 2\pi$. Then $\mathbf{r}(t)$ parameterizes $-\partial S_1$ since this standard parameterization goes around the circle in a counter-clockwise direction (this is fine – we'll fix the sign at the end). Next, we need the derivative of our parameterization: $\mathbf{r}'(t) = \langle -2\sin(t), 2\cos(t), 0 \rangle$. Now we can plug everything in.

$$\int_{-\partial S_1} \mathbf{F} \bullet d\mathbf{r} = \int_{-\partial S_1} \langle z, x, x \rangle \bullet d\mathbf{r} = \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \bullet \mathbf{r}'(t)\, dt = \int_0^{2\pi} \langle 0, 2\cos(t), 2\cos(t) \rangle \bullet \langle -2\sin(t), 2\cos(t), 0 \rangle\, dt$$
$$= \int_0^{2\pi} 4\cos^2(t)\, dt = \int_0^{2\pi} 2(1 + \cos(2t))\, dt = \int_0^{2\pi} 2 + 2\cos(2t)\, dt = 4\pi.$$

Therefore, $\displaystyle\int_{\partial S_1} \mathbf{F} \bullet d\mathbf{r} = \boxed{-4\pi}$ as we figured out before.