Math 2130-101

Test #1

June 5th, 2019

Name: Answer Key

Be sure to show your work!

1. (24 points) Vector Basics: Let $\mathbf{v} = \langle -2, 0, 1\rangle$ and $\mathbf{w} = \langle 1, 2, -2\rangle$.

(a) Compute $\text{proj}_{\mathbf{w}}(\mathbf{v}) = \dfrac{\mathbf{v}\bullet\mathbf{w}}{|\mathbf{w}|^2}\mathbf{w} = \dfrac{(-2)1+0(2)+1(-2)}{1^2+2^2+(-2)^2}\langle 1,2,-2\rangle = \boxed{\dfrac{-4}{9}\langle 1,2,-2\rangle}$.

(b) Find two *unit* vectors which are *orthogonal* (i.e., perpendicular) to both $\mathbf{v}$ and $\mathbf{w}$.

$$\mathbf{v}\times\mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 0 & 1 \\ 1 & 2 & -2 \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 2 & -2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -2 & 1 \\ 1 & -2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -2 & 0 \\ 1 & 2 \end{vmatrix}\mathbf{k} = (0(-2)-2(1))\mathbf{i} - ((-2)(-2)-1(1))\mathbf{j} + ((-2)2-1(0))\mathbf{k}$$

$= -2\mathbf{i} - 3\mathbf{j} - 4\mathbf{k} = \langle -2,-3,-4\rangle$ is perpendicular to both $\mathbf{v}$ and $\mathbf{w}$. To get a unit vector we need to normalize: $\dfrac{\mathbf{v}\times\mathbf{w}}{|\mathbf{v}\times\mathbf{w}|} = \dfrac{1}{\sqrt{(-2)^2+(-3)^2+(-4)^2}}\langle -2,-3,-4\rangle$. To get a second unit vector (perpendicular to $\mathbf{v}$ and $\mathbf{w}$) we negate.

Thus $\boxed{\pm\dfrac{\mathbf{v}\times\mathbf{w}}{|\mathbf{v}\times\mathbf{w}|} = \dfrac{\pm 1}{\sqrt{29}}\langle 2,3,4\rangle}$ give us two unit vectors perpendicular to both $\mathbf{v}$ and $\mathbf{w}$.

(c) Find the angle between $\mathbf{v}$ and $\mathbf{w}$ (don't worry about evaluating inverse trig. functions).

Recall that $\mathbf{v}\bullet\mathbf{w} = |\mathbf{v}|\,|\mathbf{w}|\cos(\theta)$ where θ is the angle between $\mathbf{v}$ and $\underset{\sim}{w}$. Solving for θ we get...

$$\theta = \arccos\left(\frac{\mathbf{v}\bullet\mathbf{w}}{|\mathbf{v}|\,|\mathbf{w}|}\right) = \arccos\left(\frac{-4}{\sqrt{5}\sqrt{9}}\right) = \boxed{\arccos\left(\frac{-4}{3\sqrt{5}}\right)}$$

Is this angle... **right,** **acute,** or [✓ **obtuse**] ? (Circle your answer.) since $\mathbf{v}\bullet\mathbf{w} = -4 < 0$ (negative).

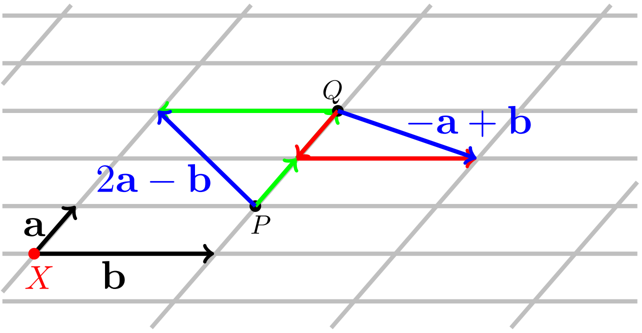
(d) Let $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ be vectors in $\mathbb{R}^3$. Fill in the blanks:

If $\mathbf{a}\times\mathbf{b} = \mathbf{0}$, then $\mathbf{a}$ and $\mathbf{b}$ are parallel.

If $\mathbf{a}\bullet\mathbf{b} = 0$, then $\mathbf{a}$ and $\mathbf{b}$ are perpendicular (or orthogonal).

If $(\mathbf{a}\times\mathbf{b})\bullet\mathbf{c} = 0$, then $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are coplanar.

(e) The vectors $\mathbf{a}$ and $\mathbf{b}$ are shown to the right. They are based at the point X. Sketch the vector $2\mathbf{a}-\mathbf{b}$ based at the point P and sketch the vector $-\mathbf{a}+\mathbf{b}$ based at the point Q.



2. (14 points) Give vector valued functions which parameterize the following curves.

Don't forget to specify a domain for your parameter.

(a) The line segment from $A = (1,2,0)$ to $B = (4,0,1)$. $\mathbf{r}(t) = A + \overrightarrow{AB}\,t = A + (B-A)t = \boxed{\langle 1,2,0\rangle + \langle 3,-2,1\rangle t, \quad 0 \le t \le 1}$

(b) The circle $(x+1)^2 + (y-3)^2 = 25$. $\boxed{\mathbf{r}(t) = \langle 5\cos(t)-1, 5\sin(t)+3\rangle, \quad 0 \le t \le 2\pi}$

Of course, there are infinitely many correct (and incorrect) answers for problem #2. For example, if we exchanged sine and cosine, we would still parameterize our above circle (centered at $(-1,3)$ with radius $\sqrt{25} = 5$). The given parameterization is the "standard" one. This traverses the circle once in a counter-clockwise direction beginning and ending at the rightmost point (i.e., $(x,y) = (-1,3)+(5,0) = (4,3)$). Interchanging sine and cosine would cause the circle to be traced out in a clockwise direction beginning and ending at its top.

Also, neglecting the domain on part (a) would yield the entire line through A and B. Neglecting the domain on part (b) would be ok (we would just trace out the circle over and over and over ...).

3. (14 points) Lines and Planes

(a) Find an equation for the plane which passes through the points $A = (1,0,-1)$, $B = (2,3,0)$, and $C = (3,-1,1)$.

We need a point and a normal vector. Any point (A, B, or C) will do. To get a normal vector we note that $\overrightarrow{AB} = B - A = \langle 1,3,1\rangle$ and $\overrightarrow{AC} = C - A = \langle 2,-1,2\rangle$ are parallel to the plane through A, B, C. Thus $\langle 1,3,1\rangle \times \langle 2,-1,2\rangle = \langle 7,0,-7\rangle$ is perpendicular to the plane (it's our desired normal vector). Using point A and our normal vector, we get $\boxed{7(x-1)+0(y-0)-7(z+1) = 0}$ (this simplifies to $x - z - 2 = 0$).

(b) Consider the line parameterized by $\mathbf{r}(t) = \langle 2t, -4t+1, 6t-2\rangle$ and the plane $-x+2y-3z=10$.

The line and plane are (circle **all** that apply)... parallel [✓ perpendicular] [✓ intersecting].

The line is parallel to $\mathbf{r}' = \langle 2,-4,6\rangle$. The plane is perpendicular to $\mathbf{n} = \langle -1,2,-3\rangle$. Notice that $\mathbf{r}' = -2\mathbf{n}$ (i.e., the line's direction is *parallel* to the plane's normal vector). Thus means that the line and plane are perpendicular (since $\mathbf{r}'$ goes with the line and $\mathbf{n}$ goes perpendicular to the plane) to each other (so they also necessarily intersect).

4. (12 points) Suppose that a particle's velocity is given by $\mathbf{v}(t) = t^2\mathbf{i} + 5e^t\mathbf{j}$. In addition, we have that this particle's initial position is $\mathbf{r}_0 = \mathbf{i}+2\mathbf{j}$. [For what it's worth, measurements are made in meters and seconds.]

(a) The particle's initial speed is $|\mathbf{v}(0)| = |\langle 0^2, 5e^0\rangle| = |\langle 0,5\rangle| = 5$ meters per second.

(b) Find the particle's acceleration $\mathbf{a}(t)$. Since $\mathbf{v}(t) = \mathbf{r}'(t)$. $\mathbf{a}(t) = \mathbf{r}''(t) = \mathbf{v}'(t) = \boxed{\langle 2t, 5e^t\rangle}$ (or $2t\mathbf{i} + 5e^t\mathbf{j}$).

(c) Find the particle's position function $\mathbf{r}(t)$. $\mathbf{r}(t) = \int \mathbf{r}'(t)\,dt = \int \mathbf{v}(t)\,dt = \int \langle t^2, 5e^t\rangle\,dt = \left\langle \frac{t^3}{3}, 5e^t\right\rangle + \mathbf{C}$. However, $\langle 1,2\rangle = \mathbf{r}_0 = \mathbf{r}(0) = \langle 0^3/3, 5e^0\rangle + \mathbf{C}$ so $\mathbf{C} = \langle 1,2\rangle - \langle 0,5\rangle = \langle 1,-3\rangle$. Thus $\boxed{\mathbf{r}(t) = \left\langle \frac{t^3}{3}+1, 5e^t-3\right\rangle}$.

5. (18 points) Let C be the curve parameterized by $\mathbf{r}(t) = \langle 3t, 4\sin(t), 4\cos(t)\rangle$, $-\pi \le t \le 3\pi$.

(a) Compute the **TNB**-frame of C. $\mathbf{r}'(t) = \langle 3, 4\cos(t), -4\sin(t)\rangle$ and $|\mathbf{r}'(t)| = \sqrt{9+16\cos^2(t)+16\sin^2(t)} = \sqrt{9+16} = 5$. Thus

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \boxed{\frac{1}{5}\langle 3, 4\cos(t), -4\sin(t)\rangle}$$

$\mathbf{T}'(t) = \frac{1}{5}\langle 0, -4\sin(t), -4\cos(t)\rangle$ and $|\mathbf{T}'(t)| = \frac{1}{5}\sqrt{0^2+16\sin^2(t)+16\cos^2(t)} = \frac{1}{5}\sqrt{16} = \frac{4}{5}$. Thus

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \frac{\frac{1}{5}\langle 0, -4\sin(t), -4\cos(t)\rangle}{4/5} = \boxed{\langle 0, -\sin(t), -\cos(t)\rangle}$$

$$\mathbf{B}(t) = \mathbf{T}(t)\times\mathbf{N}(t) = \frac{1}{5}\langle -4\cos^2(t) - 4\sin^2(t), -(-3\cos(t)), -3\sin(t)\rangle = \boxed{\frac{1}{5}\langle -4, 3\cos(t), -3\sin(t)\rangle}$$

(b) Compute the curvature of C.

If we use the *right formula* (taking into account part (a)'s calculations) this is easy: $\kappa = \frac{|\mathbf{T}'|}{|\mathbf{r}'|} = \frac{4/5}{5} = \boxed{\frac{4}{25}}$.

(c) Compute the arc length of C.

$$\text{TotalArcLength} = \int_a^b |\mathbf{r}'(t)|\,dt = \int_{-\pi}^{3\pi} 5\,dt = 5t\Big|_{-\pi}^{3\pi} = \boxed{20\pi}$$

(d) Set up (but **do not evaluate**) the line integral $\int_C x\,e^{y^2+z^2}\,ds$.

We plug in our parameterization: $x(t) = 3t$, $y(t) = 4\sin(t)$, $z(t) = 4\cos(t)$, and $ds = |\mathbf{r}'(t)|\,dt = 5\,dt$. Thus...

$$\int_C x\,e^{y^2+z^2}\,ds = \int_{-\pi}^{3\pi} 3t\,e^{16\cos^2(t)+16\sin^2(t)}5\,dt = \boxed{\int_{-\pi}^{3\pi} 15e^{16}\,t\,dt}$$

6. (18 points) Consider the curve C parameterized by $\mathbf{r}(t) = \langle t, t^2, t^3\rangle$.

(a) Find a parameterization for the line tangent to C at $t=2$.

For future reference: $\mathbf{r}'(t) = \langle 1, 2t, 3t^2\rangle$, $\mathbf{r}''(t) = \langle 0,2,6t\rangle$, and $\mathbf{r}'''(t) = \langle 0,0,6\rangle$.

Thus $\mathbf{r}(2) = \langle 2,4,8\rangle$ (point) and $\mathbf{r}'(2) = \langle 1,4,12\rangle$ (direction).

Therefore, $\boxed{\ell(t) = \langle 2,4,8\rangle + \langle 1,4,12\rangle t}$ parameterizes our tangent line.

(b) Compute the curvature of C.

Since we haven't computed this curve's TNB-frame (which would be fairly horrible), we are better off using the cross product formula for curvature.

$\mathbf{r}' \times \mathbf{r}'' = \langle 1, 2t, 3t^2\rangle \times \langle 0, 2, 6t\rangle = \langle 2t(6t) - 2(3t^2), -(1(6t) - 0(3t^2)), 1(2) - 0(2t)\rangle = \langle 6t^2, -6t, 2\rangle = 2\langle 3t^2, -3t, 1\rangle$.

$$\kappa(t) = \frac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|^3} = \boxed{\frac{2\sqrt{9t^4 + 9t^2 + 1}}{\left(1 + 4t^2 + 9t^4\right)^{3/2}}}$$

(c) Compute the torsion of C.

$$\tau(t) = \frac{(\mathbf{r}' \times \mathbf{r}'') \bullet \mathbf{r}'''}{|\mathbf{r}' \times \mathbf{r}''|^2} = \frac{\langle 6t^2, -6t, 2\rangle \bullet \langle 0, 0, 6\rangle}{36t^4 + 36t^2 + 4} = \frac{12}{36t^4 + 36t^2 + 4} = \boxed{\frac{3}{9t^4 + 9t^2 + 1}}$$

(d) Compute the tangential and normal components of acceleration of $\mathbf{r}(t)$.

$$a_{\mathbf{T}} = \frac{\mathbf{r}' \bullet \mathbf{r}''}{|\mathbf{r}'|} = \boxed{\frac{4t + 18t^3}{\sqrt{9t^4 + 4t^2 + 1}}} \qquad \text{and} \qquad a_{\mathbf{N}} = \frac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|} = \boxed{\frac{2\sqrt{9t^4 + 9t^2 + 1}}{\sqrt{9t^4 + 4t^2 + 1}}}$$

(e) Does this curve lie in a plane? Why or why not? $\boxed{\text{No, since } \tau \neq 0.}$

A smooth curve is planar if and only if its torsion is identically zero. Alternatively, if we can compute a TNB-frame, a curve is planar if and only if the binormal is constant.