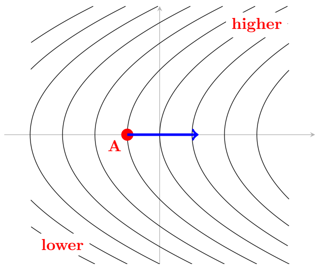
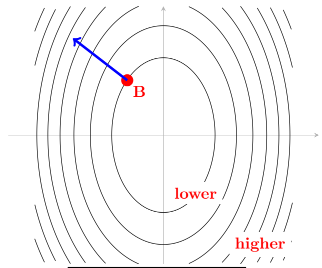
Math 2130 **Test #2** June 17th, 2019

Name: Answer Key **Be sure to show your work!**

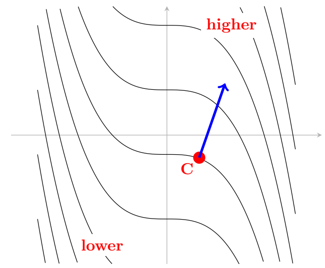
1. (12 points) Three level curve plots are shown below. I have labeled the levels so you know which curves are higher and which are lower.



$x - y^2$



$\dfrac{x^2}{2} + \dfrac{y^2}{3}$



$2x^3 + y$

(a) The plots above correspond to 3 of the 5 functions listed here: $f(x,y) = x^2/2 + y^2/3$, $f(x,y) = x^2 + y^2$, $f(x,y) = 2x^3 + y$, $f(x,y) = y - x^2$, and $f(x,y) = x - y^2$. Write the correct formula below each plot.

$x^2/2 + y^2/3 = C$ for positive constants C would be ellipses: $x^2/(2C) + y^2/(3C) = 1$ (the bigger y-term denominator, $3C$ vs. $2C$, means these ellipses should be taller than wide). Also, as C increases they would have larger "radii" (bigger ellipses correspond to higher ground). This certainly matches the middle plot. $x^2 + y^2 = C$ yields circles of radius $\sqrt{C}$ (for positive C's). This doesn't appear above. $2x^3 + y = C$ yields $y = -2x^3 + C$. These are the cubic $y = -2x^3$ (snaking downward) shifted vertically by C. This matches the third plot. $y - x^2 = C$ yields $y = x^2 + C$. These are parabolas opening upward (not pictured above). $x - y^2 = C$ yields $x = y^2 + C$. These are parabolas opening to the right (as C increases, we shift further right). This matches our first plot.

(b) Sketch a gradient vector at the points A, B, and C. If the vector is $\mathbf{0}$ or does not exist, draw an "X" on the point. [Don't worry about having the correct length. I'm just looking for the correct direction.]

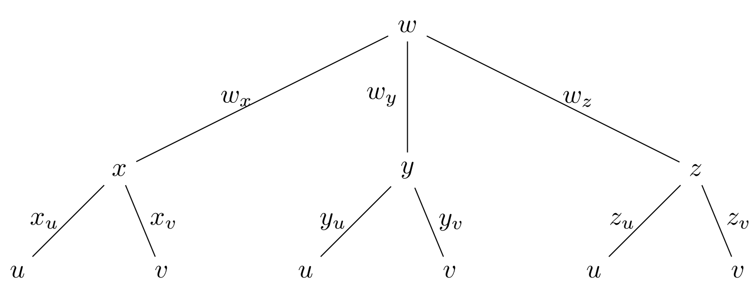
Gradient vectors point "up hill" and are perpendicular to level curves.

2. (8 points) Let $w = f(x,y,z)$, $x = g(u,v)$, $y = h(u,v)$, and $z = \ell(u,v)$.

State the chain rule for $\dfrac{\partial w}{\partial u}$.

$$\boxed{\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u}}$$

Alternate notations:
$w_u = w_x\, x_u + w_y\, y_u + w_z\, z_u$
or
$w_u = f_x\, g_u + f_y\, h_u + f_z\, \ell_u$.



3. (9 points) Suppose we have a function $f(x,y)$ where $\nabla f(x,y)$ exists everywhere.

(a) It is possible for $f_{xy}(2,3) = 4$ and $f_{yx}(2,3) = 5$? If not, why not? If so, what does this tell us?

$\boxed{\text{Yes.}}$ Clairaut's theorem says that mixed partials must be equal *if they are continuous*. Thus if $f_{xy}(2,3) = 4 \neq 5 = f_{yx}(2,3)$, then it must be that $\boxed{f_{xy} \text{ and } f_{yx} \text{ are discontinuous}}$ at $(x,y) = (2,3)$.

(b) Can I conclude $f(x,y)$ is differentiable? **YES** / $\boxed{\textbf{NO}}$

Since $\nabla f(x,y)$ exists, we have that f's first partials exist. This is not enough to guarantee that f is differentiable. If we knew that f_x and f_y were continuous, then we could conclude that f is differentiable. But existence of first partials is not enough.

(c) Can I conclude that $f(x,y)$ is continuous? **YES** / $\boxed{\textbf{NO}}$

Existence of partials is also not enough to guarantee that f is continuous. If we knew that f was differentiable, then that would be enough to guarantee f is continuous.

4. (10 points) Limits and continuity.

(a) Where is the function $f(x,y) = \ln(x^2+y^2)$ continuous?

The natural logarithm's domain is all positive reals (and it is continuous on its domain). Thus for $f(x,y)$ to be defined we need $x^2+y^2>0$. We know that $x^2+y^2 \geq 0$. Also, $x^2+y^2=0$ only if $x=y=0$. Thus the domain of $f(x,y)$ (which is where it is continuous) is everywhere $\boxed{(x,y) \neq (0,0)}$. Alternatively, we could write the domain as: $\{(x,y) \in \mathbb{R}^2 \mid (x,y) \neq (0,0)\} = \mathbb{R}^2 - \{(0,0)\}$.

(b) Show that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2xy}{x^2+y^2}$ does not exist.

We typically shoot down (multivariate) limits by getting different limits when approaching along different curves. If we approach along the y-axis: $y=0$, the limit becomes $\displaystyle\lim_{x\to 0} \frac{2x(0)}{x^2+0^2} = \lim_{x\to 0} \frac{0}{x^2} = \lim_{x\to 0} 0 = 0$.

Approaching along the x-axis would also yield 0 (this doesn't help). Notice that $y=x$ "unifies the denominator" so let's approach along that line. We get $\displaystyle\lim_{x\to 0} \frac{2x(x)}{x^2+x^2} = \lim_{x\to 0} \frac{2x^2}{2x^2} = \lim_{x\to 0} 1 = 1$.

Therefore, since approaching the origin along the y-axis yields 0 while approaching the diagonal line $y=x$ yields 1, this limit cannot exist (if it did all approaches would have to yield the same limit).

5. (14 points) Let $F(x,y,z) = y^2z^3 + e^{x^2z}$. *Note:* All three parts use the same function and point.

We need $\nabla F = \langle F_x, F_y, F_z \rangle = \langle 2xze^{x^2z}, 2yz^3, 3y^2z^2 + x^2e^{x^2z} \rangle$ for future computations.

(a) Find an equation for the plane tangent to $y^2z^3 + e^{x^2z} = -3$ at $(x,y,z) = (0,2,-1)$

The above equation is $F(x,y,z) = -3$ (i.e., we have a level surface of F). Thus $\nabla F(0,2,-1) = \langle 0, 2(2)(-1)^3, 3(2^2)(-1)^2 + 0 \rangle = \langle 0,-4,12 \rangle$ is normal to the level surface $F(x,y,z) = -3$ at $(x,y,z) = (0,2,-1)$.

The plane normal to $\langle 0,-4,12 \rangle$ and through the point $(0,2,-1)$ has equation $\boxed{0(x-0) - 4(y-2) + 12(z+1) = 0}$ (or $-4y+12z+20=0$ or $y-3z=5$).

(b) Find the directional derivative $D_{\mathbf{u}}F(0,2,-1)$ where $\mathbf{u}$ points in the same direction as $\mathbf{v} = \langle 2,-2,1 \rangle$.

We need a unit vector $\mathbf{u} = \dfrac{\mathbf{v}}{|\mathbf{v}|} = \dfrac{1}{\sqrt{2^2+(-2)^2+1^2}}\langle 2,-2,1 \rangle = \dfrac{1}{3}\langle 2,-2,1 \rangle$.

$$D_{\mathbf{u}}F(0,2,-1) = \nabla F(0,2,-1) \bullet \mathbf{u} = \langle 0,-4,12 \rangle \bullet \left(\frac{1}{3}\langle 2,-2,1 \rangle\right) = \frac{1}{3}\left(0(2) - 4(-2) + 12(1)\right) = \boxed{\frac{20}{3}}.$$

(c) Find the direction vector $\mathbf{u}$ which maximizes $D_{\mathbf{u}}F(0,2,-1)$. What is the maximum value?

Directional derivatives are maximized in the gradient direction and their maximal value is given by the magnitude of the gradient. [They are minimized the direction opposite that of the gradient and the minimal value is the magnitude of the gradient negated.]

$\mathbf{u} = \dfrac{\nabla F(0,2,-1)}{|\nabla F(0,2,-1)|} = \dfrac{\langle 0,-4,12 \rangle}{4\sqrt{0^2+(-1)^2+3^2}} = \dfrac{1}{4\sqrt{10}}\langle 0,-4,12 \rangle = \boxed{\dfrac{1}{\sqrt{10}}\langle 0,-1,3 \rangle}$. The directional derivative's maximal value at $(x,y,z) = (0,2,-1)$ is $\boxed{|\nabla F(0,2,-1)| = 4\sqrt{10}}$.

6. (8 points) Let $e^{3x}\sin(y^2z) + y\ln(x^4+z^2) = 99$. Assuming z is a function of x and y, find $\dfrac{\partial z}{\partial y}$.

The derivative of a function defined (implicitly) by $F(x,y,z) = e^{3x}\sin(y^2z) + y\ln(x^4+z^2) = 99$ is...

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = \boxed{-\frac{2yze^{3x}\cos(y^2z) + \ln(x^4+z^2)}{y^2e^{3x}\cos(y^2z) + \dfrac{2yz}{x^4+z^2}}}$$

7. (13 points) Let $f(x,y) = -x^4 + 4xy - 2y^2 - 3$.

(a) Compute the gradient and Hessian matrix for f.

$$\nabla f = \langle f_x, f_y \rangle = \boxed{\langle -4x^3+4y, 4x-4y \rangle} \qquad H_f = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \boxed{\begin{bmatrix} -12x^2 & 4 \\ 4 & -4 \end{bmatrix}}$$

(b) Find the quadratic approximation of f at $(x,y) = (0,-1)$.

We need to plug in our point: $f(0,-1) = 0+0-2(-1)^2-3 = -5$, $\nabla f(0,-1) = \langle 0+4(-1), 0-4(-1) \rangle = \langle -4,4 \rangle$, and $H_f(0,-1) = \begin{bmatrix} 0 & 4 \\ 4 & -4 \end{bmatrix}$.

$$Q(x,y) = \boxed{-5 + \langle -4,4 \rangle \bullet \langle x-0, y+1 \rangle + \frac{1}{2}\begin{bmatrix} x & y+1 \end{bmatrix}\begin{bmatrix} 0 & 4 \\ 4 & -4 \end{bmatrix}\begin{bmatrix} x \\ y+1 \end{bmatrix}}$$

or $Q(x,y) = -5 - 4x + 4(y+1) + \frac{1}{2}(0)x^2 + \frac{1}{2}(4)x(y+1) + \frac{1}{2}(4)x(y+1) + \frac{1}{2}(-4)(y+1)^2$

(c) Find and classify all of the critical points of f. [Use the "2^{nd}-derivative" test to determine if critical points are relative max's, min's or saddle points.]

To speed you along: There are exactly 3 critical points. Their x-coordinates are $x = 0, \pm 1$.

We need to solve $\nabla f = \langle -4x^3 + 4y, 4x - 4y \rangle = \mathbf{0}$. Thus $-4x^3 + 4y = 0$ and $4x - 4y = 0$ so $x^3 - y = 0$ and $x - y = 0$. Thus $y = x$ so $x^3 - x = 0$. Factoring: $x(x^2 - 1) = x(x-1)(x+1) = 0$. Thus $x = 0, \pm 1$ (as we were told). Finally $y = x$ so our 3 critical points are just $(x, y) = (0, 0)$, $(1, 1)$, and $(-1, -1)$.

To classify: $H_f(0,0) = \begin{bmatrix} 0 & 4 \\ 4 & -4 \end{bmatrix}$ thus $\det H_f(0,0) = 0(-4) - 4(4) = -16 < 0$. Therefore, $\boxed{(0,0) \text{ is a saddle point.}}$

Next, $H(\pm 1, \pm 1) = \begin{bmatrix} -12(\pm 1)^2 & 4 \\ 4 & -4 \end{bmatrix} = \begin{bmatrix} -12 & 4 \\ 4 & -4 \end{bmatrix}$ so $\det H_f(\pm 1, \pm 1) = -12(-4) - 4(4) = 32 > 0$ and $f_{xx}(\pm 1, \pm 1) = -12 < 0$. Therefore, $\boxed{(1,1) \text{ and } (-1,-1) \text{ are relative maximums.}}$

8. (14 points) Suppose $f(x, y)$ is a "nice" function (with continuous partials of all orders).

(a) $Q(x, y) = 1 + 2(x-1) + 3(x-1)(y+2) + 4(y+2)^2$ is the quadratic approx. at $(x, y) = (1, -2)$.

We can either match Q's coefficients to what they should be or we can use the fact that Q and f have matching first and second partial values at our base point $(1, -2)$. Thus differentiating Q and plugging in $(1, -2)$ can give us our answers. Reading off coefficients we have that $f(1, -2) = 1$ then...

$$\boxed{\nabla f(1,-2) = \langle 2, 0 \rangle \qquad H_f(1,-2) = \begin{bmatrix} 0 & 3 \\ 3 & 8 \end{bmatrix}}$$

Notice that there is no $(y+2)$ term so the second entry of $\nabla f(1, -2)$ is 0. Similarly the $(x-1)^2$ term is missing so $f_{xx}(1, -2) = 0$. The coefficient next to $(x-1)(y+2)$ is 3. This must account for both $\dfrac{f_{xy}(1,-2)}{2}$ and $\dfrac{f_{yx}(1,-2)}{2}$ (which should be equal since we have continuous partials). So $3 = \dfrac{f_{xy}(1,-2)}{2} + \dfrac{f_{yx}(1,-2)}{2} = f_{xy}(1, -2)$. Finally, the coefficient of $(y+2)^2$ is 4 which should match $f_{yy}(1, -2)/2$. Thus $f_{yy}(1, -2) = 8$.

Is $(x, y) = (1, -2)$ a critical point of $f(x, y)$? **YES** / $\boxed{\textbf{NO}}$

If not, why not? If so, what kind (relative min, relative max, saddle point or not enough information)?

To be a critical point the function needs to be non-differentiable or the gradient needs to be zero. However, since f has continuous partials, it must be differentiable and we found that $\boxed{\nabla f(1,-2) \neq \mathbf{0}}$. Therefore, $(1, -2)$ is not a critical point.

To get the linearization at $(1, -2)$ we just throw away the second order terms in $Q(x, y)$.

The linearization of $f(x, y)$ at $(x, y) = (1, -2)$ is $L(x, y) =$ $\underline{\quad 1 + 2(x-1) \quad}$

(b) $Q(x, y) = 12 - 5(x-2)^2 + (x-2)y - 3y^2$ is the quadratic approx. at $(x, y) = (2, 0)$.

$$\boxed{\nabla f(2,0) = \langle 0, 0 \rangle \qquad H_f(2,0) = \begin{bmatrix} -10 & 1 \\ 1 & -6 \end{bmatrix}}$$

Is $(x, y) = (2, 0)$ a critical point of $f(x, y)$? $\boxed{\textbf{YES}}$ / **NO**

If not, why not? If so, what kind (relative min, relative max, saddle point or not enough information)?

This is a critical point since $\nabla f(2, 0) = \mathbf{0}$. Notice that $\det H_f(2, 0) = -10(-6) - 1(1) = 59 > 0$ and $f_{xx}(2, 0) = -10 < 0$ so this is a $\boxed{\text{relative maximum}}$.

9. (12 points) Use the method of Lagrange multipliers to find the minimum and maximum values of

$$f(x, y) = 2x - 4y \text{ constrained to } x^2 + y^2 = 5.$$

Given $f(x, y) = 2x - 4y$ and $g(x, y) = x^2 + y^2$, $\nabla f = \langle 2, -4 \rangle = \lambda \nabla g = \lambda \langle 2x, 2y \rangle$. Thus we get the equations: $2 = 2x\lambda$, $-4 = 2y\lambda$, and $x^2 + y^2 = 5$.

Let's symmetrize the first two equations (multiply the first equation by y and the second by x): $2y = 2xy\lambda = -4x$. Therefore, $y = -2x$. Plugging this into the constraint yields $x^2 + (-2x)^2 = 5$ so that $5x^2 = 5$. Thus $x = \pm 1$. Now $y = -2x$ so $y = \mp 2$. Our two critical points are $(x, y) = (-1, 2)$ and $(1, -2)$. Plugging these into our objective function yields $f(-1, 2) = 2(-1) - 4(2) = -10$ and $f(1, -2) = 2(1) - 4(-2) = 10$.

Therefore, f subject to the constraint $x^2 + y^2 = 5$ attains a $\boxed{\text{maximum value of } 10}$ at $(x, y) = (1, -2)$ and a $\boxed{\text{minimum value of } -10}$ at $(x, y) = (-1, 2)$.