Math 251 Fall 2005 Exam #1 Answer Key

1. (10 points): Consider the function

$$f(x,y) = \begin{cases} \dfrac{x^3+y^3}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

(a) Is $f(x,y)$ continuous at the origin?
Recall the formulas for changing to polar coordinates: $x = r\cos(\theta)$ and $y = r\sin(\theta)$. Also, notice that the coordinates for the origin are $(0,0)$ in either coordinate system. Then we have that

$$\begin{aligned} \lim_{(x,y)\to(0,0)} f(x,y) &= \lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2} \\ &= \lim_{(r,\theta)\to(0,\theta)} \frac{r^3\cos^3(\theta)+r^3\sin^3(\theta)}{r^2\cos^2(\theta)+r^2\sin^2(\theta)} \\ &= \lim_{(r,\theta)\to(0,\theta)} \frac{r^2(r\cos^3(\theta)+r\sin^3(\theta))}{r^2} \\ &= \lim_{(r,\theta)\to(0,\theta)} r(\cos^3(\theta)+\sin^3(\theta)) \\ &= 0 = f(0,0) \end{aligned}$$

So we know that the function is defined at $(0,0)$, limit exists at $(0,0)$, and the limit matches the function value. Thus *YES* $f(x,y)$ is continous at $(0,0)$.

NOTE: Notice that in the limit above, we take the limit as (r,θ) goes to $(0,\theta)$ not $(0,0)$. We have to leave θ arbitrary since the origin in the polar plane does not have a well-defined angle. In particular, if we had computed this limit and found that the answer depended on θ, we would then conclude that the limit does not exist!

(b) Where is $f(x,y)$ continuous?
$\frac{x^3+y^3}{x^2+y^2}$ is built up from continuous functions and thus is continuous everywhere it is defined. The only problem happens when $x^2+y^2=0$. The only solution to this equation is $(x,y)=(0,0)$. Thus $f(x,y) = \frac{x^3+y^3}{x^2+y^2}$ is continuous for all $(x,y)\neq(0,0)$. We have all ready shown that $f(x,y)$ is continous at $(x,y) = (0,0)$. Therefore, $f(x,y)$ is *continuous everywhere* (all $\mathbb{R}^2$).

2. (10 points): Show that $\frac{\mathrm{d}}{\mathrm{d}t}[\mathbf{r}(t)\times\mathbf{r}'(t)] = \mathbf{r}(t)\times\mathbf{r}''(t)$.

Recall the product rule for the derivative of the cross product of two vector functions:

$$\frac{\mathrm{d}}{\mathrm{d}t}[\mathbf{r_1}(t)\times\mathbf{r_2}(t)] = \mathbf{r_1}'(t)\times\mathbf{r_2}(t) + \mathbf{r_1}(t)\times\mathbf{r_2}'(t)$$

Therefore, we have that

$$\frac{\mathrm{d}}{\mathrm{d}t}[\mathbf{r}(t)\times\mathbf{r}'(t)] = \mathbf{r}'(t)\times\mathbf{r}'(t) + \mathbf{r}(t)\times\mathbf{r}''(t) = 0 + \mathbf{r}(t)\times\mathbf{r}''(t)$$

because $\mathbf{v}\times\mathbf{v} = 0$ for any vector $\mathbf{v}$.

3. (12 points): Let $f(x,y) = xe^{xy}$.

(a) Find $\mathbf{D_u} f$ at $(x,y) = (1,0)$ where $\mathbf{u} = \langle \frac{3}{5}, \frac{4}{5} \rangle$.
First, let's compute the gradient of f at $(1,0)$. $\nabla f(x,y) = \langle f_x, f_y \rangle = \langle e^{xy} + xye^{xy}, x^2 e^{xy} \rangle$. Thus the gradient at $(1,0)$ is $\nabla f(1,0) = \langle e^0 + 1(0)e^0, 1^2 e^0 \rangle = \langle 1, 1 \rangle$. Therefore, $\mathbf{D_u} f(1,0) = \nabla f(1,0) \cdot \langle \frac{3}{5}, \frac{4}{5} \rangle = \langle 1, 1 \rangle \cdot \langle \frac{3}{5}, \frac{4}{5} \rangle = \frac{3}{5} + \frac{4}{5} = \frac{7}{5}$.

(b) What is the maximal value of $\mathbf{D_u} f$ at $(x,y) = (1,0)$?
The maximal directional derivative at a point occurs when $\mathbf{u}$ is the unit vector parallel to the gradient vector for that point. It's maximal value is the length of the gradient at that point. Thus the maximal value is $|\nabla f(1,0)| = |\langle 1,1 \rangle| = \sqrt{2}$.

(c) Find $\mathbf{u}$ such that $\mathbf{D_u} f$ takes on this maximal value at $(x,y) = (1,0)$.
As discussed in part (b),

$$\mathbf{u} = \frac{\nabla f(1,0)}{|\nabla f(1,0)|} = \left\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right\rangle.$$

4. (10 points): Let $\ln(x + 2y + 3z) = 4$. Find $\dfrac{\partial z}{\partial x}$.

Notice that we have something of the form $F(x,y,z) = K$ (constant). Computing partials of F, we find that

$$F_x = \frac{1}{x+2y+3z} \qquad \text{and} \qquad F_z = \frac{3}{x+2y+3z}$$

Therefore, using the formula sheet we have that

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{\frac{1}{x+2y+3z}}{\frac{3}{x+2y+3z}} = -\frac{1}{3}$$

5. (12 points): Consider the paraboloid $z = x^2 + y^2$.

(a) Find the tangent plane at the point $(1,-1,2)$.
There are several formulas we could use. Let's just use the gradient. $z = x^2 + y^2$ is the same as the level surface $F(x,y,z) = x^2 + y^2 - z = 0$. $\nabla F(x,y,z) = \langle 2x, 2y, -1 \rangle$ and thus $\nabla F(1,-1,2) = \langle 2,-2,-1 \rangle$. We know that the gradient vector is perpendicular to every tangent vector to the surface at that point. Thus we can use $\langle 2,-2,-1 \rangle$ as the normal vector for our plane. The tangent plane to the surface at $(1,-1,2)$ must actually contain the point $(1,-1,2)$. Therefore, we have that

$$2(x-1) + (-2)(y - (-1)) + (-1)(z-2) = 0$$

is the equation of the tangent plane at the point $(1,-1,2)$. This simplifies to $z = 2x - 2y - 2$.

(b) Find parametric equations for the normal line at the point $(1, -1, 2)$.
The normal line has the same direction is the normal vector of the tangent plane. The normal line at the point $(1, -1, 2)$ must also contain the point $(1, -1, 2)$. Thus we have the parametric equations:

$$\begin{aligned} x(t) &= 2t + 1 \\ y(t) &= -2t - 1 \\ z(t) &= -t + 2 \end{aligned}$$

Note: Different parametrizations give different looking but equally correct answers.

6. (12 points):

(a) Find parametric equations for the line through the points $(-4, -6, 1)$ and $(-2, 0, -3)$.
To write parametric equations for the line, we need a point (either will do) and a direction vector. We get the direction vector by subtracting the coordinates of one point from the other. So we get $\langle -4 - (-2), -6 - 0, 1 - (-3) \rangle = \langle -2, -6, 4 \rangle$ OR $\langle -2 - (-4), 0 - (-6), -3 - 1 \rangle = \langle 2, 6, -4 \rangle$. Either vector will work. Let's use the second vector and the second point. We get:

$$\begin{aligned} x(t) &= 2t - 2 \\ y(t) &= 6t \\ z(t) &= -4t - 3 \end{aligned}$$

Note: Different parametrizations give different looking but equally correct answers.

(b) Find parametric equations for the line through the points $(10, 18, 4)$ and $(5, 3, 14)$.
Same game as part (a). Let's use the point $(5, 3, 14)$ and the direction vector $\langle 10 - 5, 18 - 3, 4 - 14 \rangle = \langle 5, 15, -10 \rangle$. We get:

$$\begin{aligned} x(t) &= 5t + 5 \\ y(t) &= 15t + 3 \\ z(t) &= -10t + 14 \end{aligned}$$

(c) Are these lines parallel? Why? or Why not?

$$\frac{5}{2}\langle 2, 6, -4 \rangle = \langle 5, 15, -10 \rangle$$

Thus the direction vectors are parallel. So *YES* these lines are parallel. Another way to see that the direction vectors are parallel is to compute their cross product – the cross product of these vectors is the zero vector.

7. (12 points): Find the curvature of $\mathbf{r}(t) = \left\langle \frac{t^5}{5}, \frac{2t^3}{3}, t+1 \right\rangle$ when $t = 1$.

First, we compute $\mathbf{r}'(t) = \langle t^4, 2t^2, 1\rangle$. Next, $\mathbf{r}''(t) = \langle 4t^3, 4t, 0\rangle$. Now that derivatives have been computed, we can plug in $t = 1$. We get: $\mathbf{r}'(1) = \langle 1, 2, 1\rangle$ and $\mathbf{r}''(1) = \langle 4, 4, 0\rangle$. Now we must compute their cross product

$$\begin{aligned}\mathbf{r}'(1) \times \mathbf{r}''(1) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 1 \\ 4 & 4 & 0 \end{vmatrix} \\ &= (2(0) - 4(1))\mathbf{i} - (1(0) - 4(1))\mathbf{j} + (1(4) - 4(2))\mathbf{k} \\ &= -4\mathbf{i} + 4\mathbf{j} - 4\mathbf{k}\end{aligned}$$

Thus, $|\mathbf{r}'(1) \times \mathbf{r}''(1)| = |\langle -4, 4, -4\rangle| = \sqrt{2^4\,3}$ and $|\mathbf{r}'(1)| = |\langle 1, 2, 1\rangle| = \sqrt{6}$. Therefore,

$$\kappa = \frac{|\mathbf{r}'(1) \times \mathbf{r}''(1)|}{|\mathbf{r}'(1)|^3} = \frac{\sqrt{2^4\,3}}{\sqrt{2^3\,3^3}} = \frac{\sqrt{2}}{3}$$

8. (10 points): Let $z = f(x, y)$, $x(t) = e^t$, and $y(t) = e^{2t}$. Find $\frac{\mathrm{d}z}{\mathrm{d}t}$.

Notice that $x'(t) = e^t$ and $y'(t) = 2e^{2t}$. Then use the chain rule to get:

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\partial z}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial z}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} = f_x(x, y)e^t + f_y(x, y)2e^{2t}$$

9. (12 points): Consider the curve $\mathbf{r}(t) = \left\langle 2t, t^2, \frac{t^3}{3} \right\rangle$. Find $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$ when $t = 0$.

First, we compute $\mathbf{r}'(t) = \langle 2, 2t, t^2\rangle$ and $|\mathbf{r}'(t)| = \sqrt{t^4 + 4t^2 + 4} = \sqrt{(t^2 + 2)^2} = t^2 + 2$.

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{\langle 2, 2t, t^2\rangle}{t^2 + 2} = \left\langle \frac{2}{t^2 + 2}, \frac{2t}{t^2 + 2}, \frac{t^2}{t^2 + 2} \right\rangle$$

So we find that $\mathbf{T}(0) = \langle 1, 0, 0\rangle$. Next, we compute $\mathbf{T}'(t)$.

$$\mathbf{T}'(t) = \left\langle \frac{0(t^2 + 2) - 2(2t)}{(t^2 + 2)^2}, \frac{2(t^2 + 2) - 2t(2t)}{(t^2 + 2)^2}, \frac{2t(t^2 + 2) - t^2(2t)}{(t^2 + 2)^2} \right\rangle$$

Thus $\mathbf{T}'(0) = \langle 0, 1, 0\rangle$.

$$\mathbf{N}(0) = \frac{\mathbf{T}'(0)}{|\mathbf{T}'(0)|} = \frac{\langle 0, 1, 0\rangle}{1} = \langle 0, 1, 0\rangle$$

Finally, $\mathbf{B}(0) = \mathbf{T}(0) \times \mathbf{N}(0) = \mathbf{i} \times \mathbf{j} = \mathbf{k}$. To sum up $\mathbf{T}(0) = \mathbf{i}$, $\mathbf{N}(0) = \mathbf{j}$, and $\mathbf{B}(0) = \mathbf{k}$.