Math 251 Fall 2005 Exam #2 Answer Key

(12 points): Let $\mathbf{F}(x, y, z) = (2x+yz)\mathbf{i}+(z\cos(y)+xz)\mathbf{j}+(\sin(y)+xy-3z^2)\mathbf{k}$. Show that $\mathbf{F}$ is conservative by finding a function f such that $\mathbf{F} = \nabla f$.

Integrate each component with respect to the proper variable ($\mathbf{i}$ goes with x, $\mathbf{j}$ goes with y, and $\mathbf{k}$ goes with z).

$$\int 2x + yz\, dx = x^2 + xyz + g_1(y, z)$$

$$\int z\cos(y) + xz\, dy = z\sin(y) + xyz + g_2(x, z)$$

$$\int \sin(y) + xy - 3z^2\, dz = z\sin(y) + xyz - z^3 + g_3(x, y)$$

Putting this together we see that $f(x, y, z) = x^2 + xyz + z\sin(y) - z^3$ (plus an arbitrary constant C).

(10 points): Let $f(x, y) = x^2 + 2y^2 + xy^2 + 1$. Find all critical points, then determine whether each critical point is a minimum, a maximum, a saddle point, or a nothing.

The first partials: $f_x = 2x + y^2$, $f_y = 4y + 2xy$. We must solve the system of equations: $f_x = 0$ and $f_y = 0$ to find the critical points. $2x + y^2 = 0$ implies that $2x = -y^2$. We can substitute this into the second equation and get $4y + (-y^2)y = 0$. This means that $y(4 - y^2) = 0$. Thus $y = 0$ or $y = \pm 2$. Now we know that $2x = -y^2$ which says that $x = -\frac{1}{2}y^2$. So if $y = 0$, then $x = 0$. And if $y = \pm 2$, then $x = -2$. Therefore, we have three critical points $(0, 0)$, $(-2, -2)$, and $(-2, 2)$.

Let's test these points. First, we must compute the second partials. $f_{xx} = 2$, $f_{xy} = f_{yx} = 2y$, and $f_{yy} = 4 + 2x$. Thus $D = f_{xx}f_{yy} - f_{xy}f_{yx} = 2(4 + 2x) - (2y)^2 = 8 + 4x - 4y^2$. $D(0, 0) = 8 > 0$ and $f_{xx}(0, 0) = 2 > 0$, so $(0, 0)$ is a relative minimum. $D(-2, -2) = 8-8-16 < 0$, so $(-2, -2)$ is a saddle point. $D(-2, 2) = 8-8-16 < 0$, so $(-2, 2)$ is a saddle point.

(12 points): Use the method of Lagrange multipliers to find the maximum and minimum of $f(x, y) = xy$ subject to the constraint $x^2 + y^2 = 4$. **WARNING:** You must use Lagrange multipliers!

The gradient of f is $y\mathbf{i} + x\mathbf{j}$ and the gradient of $x^2 + y^2$ is $2x\mathbf{i} + 2y\mathbf{j}$ Thus we have the equations $y = 2x\lambda$ and $x = 2y\lambda$. Suppose that x and y are non-zero. Then:

$$\frac{y}{x} = 2\lambda = \frac{x}{y}$$

Therefore, $x^2 = y^2$. Now using the constraint equation we get that $x^2+y^2 = 2x^2 = 4$ Thus $x = \pm\sqrt{2}$. Now $x^2 = y^2$, so $y = \pm x$. If $x = \sqrt{2}$, then $y = \pm\sqrt{2}$. If $x = -\sqrt{2}$, then $y = \pm\sqrt{2}$. So we have four solutions $(\sqrt{2}, \sqrt{2})$, $(\sqrt{2}, -\sqrt{2})$, $(-\sqrt{2}, \sqrt{2})$, and $(-\sqrt{2}, -\sqrt{2})$.

Next notice that $f(-\sqrt{2}, -\sqrt{2}) = f(\sqrt{2}, \sqrt{2}) = 2$ and $f(\sqrt{2}, -\sqrt{2}) = f(-\sqrt{2}, \sqrt{2}) = -2$.

If either x or y is zero, then their product is zero (this also forces $\lambda = 0$). So we ignore those points.

The maximum value of f is 2. The minimum value of f is -2.

(12 points): Evaluate the following integral using an *obvious* change of variables:

$$\iint_R \frac{2x+y}{x-y}\,dA$$

where R is the parallelogram bounded by the lines $2x+y=1$, $2x+y=0$, $x-y=2$, and $x-y=1$.

Notice that both $2x+y$ and $x-y$ appear in the integral and in the bounds. So we choose a (linear) change of variables: $u = 2x+y$ and $v = x-y$. Now the bounds change to $u=1$, $u=0$, $v=2$, and $v=1$. So we will be integrating over a rectangle (*much better*).

Next we need to compute the Jacobian. One way to do this is to solve for x and y in the change of variables equations. Let's add the equations together. We get $u+v = 2x+y+x-y$ $(=3x)$. Thus $x = \frac{1}{3}u + \frac{1}{3}v$. Next, we'll eliminate x. Let's subtract 2 times the second equation from the first equation. We get $u - 2v = 2x+y-2x+2y$ $(=3y)$. Thus $y = \frac{1}{3}u - \frac{2}{3}v$. Let's compute the Jacobian:

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} \end{vmatrix} = -\frac{2}{9} - \frac{1}{9} = -\frac{1}{3}$$

Therefore,

$$\iint_R \frac{2x+y}{x-y}\,dA = \int_1^2 \int_0^1 \frac{u}{v}\left|-\frac{1}{3}\right| du\,dv = \int_1^2 \frac{1}{3v}\frac{1}{2}u^2\Big|_0^1 dv = \int_1^2 \frac{1}{6v}\,dv = \frac{1}{6}\ln(v)\Big|_1^2 = \frac{1}{6}\ln(2)$$

Note: We could have saved ourselves some work by computing the Jacobian of the inverse transformation:

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} = -2 - 1 = -3$$

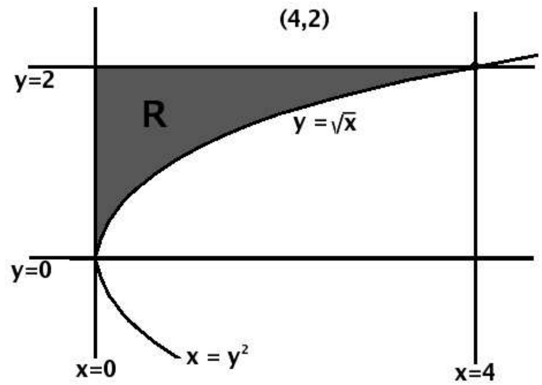
Thus

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\frac{\partial(u,v)}{\partial(x,y)}} = -\frac{1}{3}$$

(11 points): Evaluate the following integral (*Hint:* reverse the order of integration):

$$\int_0^4 \int_{\sqrt{x}}^2 3\sqrt{1+y^3}\,dy\,dx$$

This region is bounded by $y = \sqrt{x}$, $y = 2$, $x = 0$, and $x = 4$. Alternatively, it is bounded by $x = 0$, $x = y^2$, $y = 0$, and $y = \sqrt{4} = 2$.



$$\int_0^4 \int_{\sqrt{x}}^2 3\sqrt{1+y^3}\, dy\, dx = \int_0^2 \int_0^{y^2} 3\sqrt{1+y^3}\, dx\, dy = \int_0^2 3x\sqrt{1+y^3}\Big|_0^{y^2} dy = \int_0^2 3y^2\sqrt{1+y^3}\, dy$$

Use the substitution $u = 1 + y^3$ ($du = 3y^2\, dy$, the bounds change to $u = 1 + 0^3 = 1$ and $u = 1 + 2^3 = 9$) and get:

$$= \int_1^9 \sqrt{u}\, du = \frac{2}{3}u^{3/2}\Big|_1^9 = \frac{2}{3}9^{3/2} - \frac{2}{3}1^{3/2} = 18 - \frac{2}{3} = \frac{52}{3}$$

(11 points): Consider the solid E which is bounded by $z = \sqrt{9 - x^2 - y^2}$ and the xy-plane. Write a triple integral which computes the volume of E in: rectangular, cylidrical, and spherical coordinates. Then find the volume of E. (*Hint:* If you identify this solid, you can use simple highschool geometry to find its volume.)

$z = \sqrt{9 - x^2 - y^2}$ intersects the xy-plane ($z = 0$) on the curve $0 = \sqrt{9 - x^2 - y^2}$ which is the circle $x^2 + y^2 = 9$. We must integrate over the inside of this circle, so we have the bounds $y = \pm\sqrt{9 - x^2}$ and $x = \pm 3$. Therefore, in rectangular cooridinates:

$$\text{Volume}(E) = \iiint_E dV = \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_0^{\sqrt{9-x^2-y^2}} dz\, dy\, dx$$

Switching to cylindrical coordinates, we need to describe the inside of the circle $r^2 = x^2 + y^2 = 9$ in polar cooridinates. But this is easy. We have the bounds $r = 0$, $r = 3$, $\theta = 0$ and $\theta = 2\pi$. Also, we need to describe the upper z bound in the new coordinates, but again this is easy $z = \sqrt{9 - x^2 - y^2} = \sqrt{9 - r^2}$. Therefore, in cylindrical coordinates:

$$\text{Volume}(E) = \iiint_E dV = \int_0^{2\pi} \int_0^3 \int_0^{\sqrt{9-r^2}} r\, dz\, dr\, d\theta$$

Finally, switching to spherical coordinates, we notice that $z = \sqrt{9-x^2-y^2}$ is really just the top cap of a sphere (thus ϕ should only range from 0 to $\pi/2$), so we get:

$$\text{Volume}(E) = \iiint_E dV = \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_0^3 \rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta$$

The equation for the volume of a sphere is $\frac{4}{3}\pi r^3$. Thus the sphere of radius 3 has volume 36π. So our solid has volume 18π (all three of the integral evaluate to 18π).

(11 points): Evaluate the following integral (*Hint:* change coordinates!):

$$\int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^{\sqrt{4-x^2-y^2}} \frac{e^{\sqrt{x^2+y^2+z^2}}}{x^2+y^2+z^2}\, dz\, dy\, dx$$

We know that $x^2+y^2+z^2=\rho^2$. So the natural choice is sphereical coordinates. The top z bound is equation of the top half of a sphere of radius 2. The x and y bounds enclose a quarter circle or radius 2. Thus we have $0 \le \rho \le 2$, $0 \le \phi \le \frac{\pi}{2}$ (just the top half of the sphere), and $0 \le \theta \le \frac{\pi}{2}$ (one quarter of the circle). Therefore, the integral transforms to:

$$\begin{aligned}
&= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^2 \frac{e^\rho}{\rho^2} \rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta \\
&= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^2 e^\rho \sin(\phi)\, d\rho\, d\phi\, d\theta \\
&= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} e^\rho \sin(\phi)\Big|_0^2\, d\phi\, d\theta \\
&= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} (e^2-1)\sin(\phi)\, d\phi\, d\theta \\
&= \int_0^{\frac{\pi}{2}} -(e^2-1)\cos(\phi)\Big|_0^{\frac{\pi}{2}}\, d\theta \\
&= \int_0^{\frac{\pi}{2}} 0 + (e^2-1)\, d\theta \\
&= (e^2-1)\theta\Big|_0^{\frac{\pi}{2}} = \frac{\pi}{2}(e^2-1)
\end{aligned}$$

(10 points): Let $\mathbf{F}(x,y) = e^{-y}\mathbf{i} + (2y - xe^{-y})\mathbf{j}$. Compute $\int_C \mathbf{F}\cdot d\mathbf{r}$ where C is any curve from $(0,0)$ to $(1,2)$.

We can see that $\frac{\partial}{\partial y}(e^{-y}) = -e^{-y} = \frac{\partial}{\partial x}(2y - xe^{-y})$. Thus $\mathbf{F}$ is conservative. Let's integrate to find a potential function.

$$\int e^{-y}\, dx = xe^{-y} + g_1(y)$$

$$\int 2y - xe^{-y}\,dy = y^2 + xe^{-y} + g_2(x)$$

Therefore, $f(x,y) = y^2 + xe^{-y}$ (plus a constant if we wish). By the fundamental theorem of line integrals:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = f(1,2) - f(0,0) = (2^2 + (1)e^{-2}) - (0+0) = 4 + e^{-2}$$

(11 points): Find the center of mass of a thin wire with constant density $\rho(x,y,z) = 2$ in the shape of the upper half of the circle of radius 2 centered at the origin.

First, we need to parametrize the circle. $\mathbf{r}(t) = 2\cos(t)\mathbf{i} + 2\sin(t)\mathbf{j}$ parametrizes the circle of radius 2 centered at the origin. If we restrict t to the interval $[0,\pi]$, we will get the top half of the circle.

Notice that $\mathbf{r}'(t) = -2\sin(t)\mathbf{i} + 2\cos(t)\mathbf{j}$. Thus $|\mathbf{r}'(t)| = \sqrt{4\sin^2(t) + 4\cos^2(t)} = 2$.

$$m = \int_C \rho(x,y)\,ds = \int_0^\pi \rho(x,y)|\mathbf{r}'(t)|\,dt = \int_0^\pi 2(2)\,dt = 4\pi$$

$$\begin{aligned} M_y &= \int_C x\rho(x,y)\,ds = \int_0^\pi x(t)\rho(x,y)|\mathbf{r}'(t)|\,dt \\ &= \int_0^\pi 2\cos(t)2(2)\,dt = \int_0^\pi 8\cos(t) = 8\sin(\pi) - 8\sin(0) = 0 \end{aligned}$$

(Or you could just use common sense to see that this must be zero – the wire is symmetric about the y-axis!)

$$\begin{aligned} M_x &= \int_C y\rho(x,y)\,ds = \int_0^\pi y(t)\rho(x,y)|\mathbf{r}'(t)|\,dt \\ &= \int_0^\pi 2\sin(t)2(2)\,dt = \int_0^\pi 8\sin(t) = -8\cos(\pi) + 8\cos(0) = 16 \end{aligned}$$

Therefore,

$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{m}, \frac{M_x}{m}\right) = \left(\frac{0}{4\pi}, \frac{16}{4\pi}\right) = \left(0, \frac{4}{\pi}\right)$$