Math 251 Spring 2007
Exam #1 Answer Key

1. (11 points): Consider the function

$$f(x,y) = \begin{cases} \dfrac{x^2+y^2}{x^2+4y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

(a) Does $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$ exist? If so, find it. If not, show that it does not exist.

We know that if the limit at $(0,0)$ exists, then approaching $(0,0)$ using any continuous curve will give the same answer. Keeping this in mind, consider the following:

Using the line $y = 0$: $$\lim_{(x,0)\to(0,0)} f(x,0) = \lim_{x\to 0} \frac{x^2+0^2}{x^2+4\cdot 0^2} = \lim_{x\to 0} \frac{x^2}{x^2} = 1$$

Using the line $x = 0$: $$\lim_{(0,y)\to(0,0)} f(0,y) = \lim_{y\to 0} \frac{0^2+y^2}{0^2+4y^2} = \lim_{y\to 0} \frac{y^2}{4y^2} = \frac{1}{4}$$

Seeing that the limit approaching the origin along the line $y = 0$ gives a different answer than when approaching the origin along the line $x = 0$, we conclude that **the limit does not exist**.

(b) Is $f(x,y)$ continuous at the point $(0,0)$? Why? or Why not?

"**No**" The limit of $f(x,y)$ as (x,y) approaches $(0,0)$ does not exist. Therefore (by definition) the function cannot be continuous at the origin.

(c) Is $f(x,y)$ continuous at the point $(1,1)$? Why? or Why not?

"**Yes**" Since $x^2+4y^2 \neq 0$ when $(x,y) = (1,1)$ we can compute the limit directly. We get:

$$\lim_{(x,0)\to(1,1)} f(x,y) = \lim_{(x,y)\to(1,1)} \frac{x^2+y^2}{x^2+4y^2} = \frac{1^2+1^2}{1^2+4\cdot 1^2} = \frac{1}{5} = f(1,1)$$

Thus (by definition) $f(x,y)$ is continous at $(1,1)$.

2. (12 points): Consider the following lines:

$$\text{Line } L_1: \begin{array}{rcrcr} x(t) & = & 1 & + & t \\ y(t) & = & 2 & - & t \\ z(t) & = & 3 & + & t \end{array} \qquad \text{Line } L_2: \begin{array}{rcrcr} x(t) & = & & & 3t \\ y(t) & = & -2 & + & 2t \\ z(t) & = & 4 & + & t \end{array}$$

(a) Are L_1 and L_2 parallel, intersecting, or skew lines?

The line L_1 is parallel to the vector $\langle 1,-1,1\rangle$ whereas L_2 is parallel to the vector $\langle 3,2,1\rangle$. Notice that $\langle 1,-1,1\rangle \neq k\langle 3,2,1\rangle$ for any k. Thus the vectors are not parallel. Therefore, L_1 and L_2 are not parallel lines.

Next, let's check if they intersect. [**WARNING:** Remember to use different parameters for different lines – they may cross at "different times".]

$$\begin{aligned} 1+t &= 3s \\ 2-t &= -2+2s \\ 3+t &= 4+s \end{aligned}$$

Using the last equation, we see that $t = 1+s$. Plugging this into the first equation we get that $1+(1+s) = 3s$. Thus $2 = 2s$ and so $s = 1$ (and thus $t = 2$). If we plug $t = 2$ into the equations for line L_1, we get the point $(3, 0, 5)$. If we plug $s = 1$ into the equations for line L_2, we get the same. Therefore, the lines L_1 and L_2 **intersect** at the point $(3, 0, 5)$.

(b) Find a plane which contains the line L_1 and is parallel to the line L_2.

Note: We know from part (a) that the lines intersect. Therefore, our plane will contain both of the lines!

The plane must be parallel to both lines. Thus it should be parallel to the vectors: $\langle 1, -1, 1\rangle$ and $\langle 3, 2, 1\rangle$. We can find a normal vector for our plane by computing the cross product of these two vectors (we all ready know they aren't parallel from part (a)).

$$\langle 1,-1,1\rangle \times \langle 3,2,1\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 1 \\ 3 & 2 & 1 \end{vmatrix} = (-1(1)-2(1))\mathbf{i}-(1(1)-3(1))\mathbf{j}+(1(2)-3(-1))\mathbf{k} = \langle -3,2,5\rangle$$

We want a plane which contains the line L_1. Since L_1 passes through the point $(1, 2, 3)$, our plane must pass through $(1, 2, 3)$ also. Therefore, we have the following equation for our plane:

$$-3(x-1) + 2(y-2) + 5(z-3) = 0$$

Note: Of course this is not the only way to get the equation of this plane. Your answer might look different if you fit the plane through a different point or used some different multiple of the normal vector.

3. (10 points): Consider the curve $\mathbf{r}(t) = \langle e^t\cos(t), \sqrt{2}e^t, e^t\sin(t)\rangle$ where $-\pi \le t \le \pi$. Find parametric equations for the tangent line of $\mathbf{r}(t)$ at $t = 0$.

First, differentiate $\mathbf{r}(t)$ and plug in $t = 0$ to find a tangent vector.
$\mathbf{r}'(t) = \langle e^t\cos(t) - e^t\sin(t), \sqrt{2}e^t, e^t\sin(t) + e^t\cos(t)\rangle$ and thus
$\mathbf{r}'(0) = \langle e^0\cos(0) - e^0\sin(0), \sqrt{2}e^0, e^0\sin(0) + e^0\cos(0)\rangle = \langle 1, \sqrt{2}, 1\rangle$.

We have a direction vector for our line. Next, we need a point to fit the line through. Plugging $t = 0$ into $\mathbf{r}(t)$ we find that $\mathbf{r}(0) = \langle e^0\cos(0), \sqrt{2}e^0, e^0\sin(0)\rangle = \langle 1, \sqrt{2}, 0\rangle$.

Therefore, our tangent line is:

$$\begin{aligned} x(t) &= 1 + t \\ y(t) &= \sqrt{2} + \sqrt{2}t \\ z(t) &= t \end{aligned}$$

4. (10 points): Let $z = f(x, y)$ where $x = s + t$ and $y = s - t$. Assuming that f is differentiable, show that:

$$\left(\frac{\partial z}{\partial x}\right)^2 - \left(\frac{\partial z}{\partial y}\right)^2 = \frac{\partial z}{\partial s} \cdot \frac{\partial z}{\partial t}.$$

Hint: Compute $\dfrac{\partial z}{\partial s}$ and $\dfrac{\partial z}{\partial t}$ using the chain rule first.

Notice that:

$$\frac{\partial x}{\partial s} = 1, \qquad \frac{\partial x}{\partial t} = 1, \qquad \frac{\partial y}{\partial s} = 1, \qquad \text{and} \qquad \frac{\partial y}{\partial t} = -1$$

According to the chain rule we have that:

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial s} = \frac{\partial z}{\partial x}(1) + \frac{\partial z}{\partial y}(1)$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial t} = \frac{\partial z}{\partial x}(1) + \frac{\partial z}{\partial y}(-1)$$

Therefore,

$$\frac{\partial z}{\partial s} \cdot \frac{\partial z}{\partial t} = \left(\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\right)\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right) = \left(\frac{\partial z}{\partial x}\right)^2 - \left(\frac{\partial z}{\partial y}\right)^2$$

5. (10 points): Random dot and cross product questions.

(a) If $\mathbf{a}$ is a unit vector in $\mathbb{R}^2$ and $\mathbf{a}\cdot\langle 1, 0\rangle = 0$, what could $\mathbf{a}$ be? Is there more than one answer?

If $\mathbf{a}\cdot\mathbf{i} = 0$, then $\mathbf{a}$ is perpendicular to the x-axis (and thus is parallel to the y-axis). Being a unit vector, we must have that $\mathbf{a} = \pm\mathbf{j} = \langle 0, \pm 1\rangle$. Thus we have **two** answers.

We could have done this problem algebraically: Let $\mathbf{a} = \langle a_1, a_2\rangle$. We have that $\mathbf{a}\cdot\langle 1, 0\rangle = a_1 = 0$. But $|\mathbf{a}| = \sqrt{a_1^2 + a_2^2} = \sqrt{a_2^2} = |a_2| = 1$ since $\mathbf{a}$ is a unit vector. Thus $a_2 = \pm 1$ and hence $\mathbf{a} = \langle 0, \pm 1\rangle$.

(b) Let $\mathbf{a}$ and $\mathbf{b}$ be vectors in $\mathbb{R}^3$ such that $\mathbf{a} \times \mathbf{b} = \mathbf{0}$. What can we say about $\mathbf{a}$ and $\mathbf{b}$?

If the cross product of $\mathbf{a}$ and $\mathbf{b}$ is zero, then either one (or both) of the vectors is the zero vector or both are non-zero vectors and $0 = |\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin(\theta)$ where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$. Thus $\sin(\theta) = 0$ and we conclude that $\theta = 0$ or π (radians).

Briefly, this means that $\mathbf{a}$ and $\mathbf{b}$ are **parallel**.

6. (10 points): Find the plane tangent to the level surface $x^2 - y^2 + z^2 = 1$ at the point $(-1, 0, 0)$.

Note: $(-1)^2 - 0^2 + 0^2 = 1$ so our point *really* does lie on the level surface.

Let $F(x, y, z) = x^2 - y^2 + z^2$. We know that $\nabla F(-1, 0, 0)$ will give us a normal vector for our tangent plane. So we compute $\nabla F = \langle 2x, -2y, 2z\rangle$. Thus $\nabla F(-1, 0, 0) = \langle -2, 0, 0\rangle$.

Our plane is tangent to the point $(-1, 0, 0)$. So we get that: $-2(x - (-1)) + 0(y - 0) + 0(z - 0) = 0$. This means that $-2(x + 1) = 0$.

$$\text{Our plane is :} \qquad x = -1.$$

7. (15 points): Consider the curve: $\mathbf{r}(t) = \langle 5\cos(t), 5\sin(t), 0\rangle$ where $0 \le t \le 2\pi$.

(a) Find a formula for the arc length function $s(t)$ of $\mathbf{r}(t)$ and then use it to reparametrize $\mathbf{r}(t)$ with respect to arc length.

By definition $s(t) = \displaystyle\int_0^t |\mathbf{r}'(u)|\,du$. We compute $\mathbf{r}'(t) = \langle -5\sin(t), 5\cos(t), 0\rangle$. Therefore, $|\mathbf{r}'(t)| = \sqrt{25\sin^2(t) + 25\cos^2(t) + 0} = \sqrt{25} = 5$. So we get that $s(t) = \displaystyle\int_0^t 5\,du = 5t$.

Next, we can easily solve $s = 5t$ for t (i.e. $t = s/5$). Plugging this into $\mathbf{r}(t)$ we find that:

$$\mathbf{r}(s) = \left\langle 5\cos\left(\frac{s}{5}\right), 5\sin\left(\frac{s}{5}\right), 0\right\rangle$$

(b) Find $\mathbf{T}(t)$, $\mathbf{N}(t)$, and $\mathbf{B}(t)$ for this curve.

Using our computations of $\mathbf{r}'(t)$ and $|\mathbf{r}'(t)|$ from part (a), we find that:

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{1}{5}\langle -5\sin(t), 5\cos(t), 0\rangle = \langle -\sin(t), \cos(t), 0\rangle.$$

Next, compute $\mathbf{T}'(t) = \langle -\cos(t), -\sin(t), 0\rangle$. Then $|\mathbf{T}'(t)| = \sqrt{\cos^2(t) + \sin^2(t)} = 1$. Therefore,

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \langle -\cos(t), -\sin(t), 0\rangle$$

Finally, we compute the binormal using our formulas for $\mathbf{T}(t)$ and $\mathbf{N}(t)$.

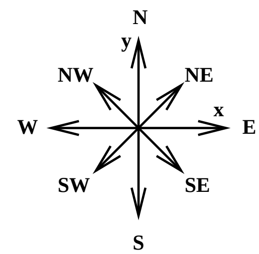
$$\begin{aligned}
\mathbf{B}(t) &= \mathbf{T}(t) \times \mathbf{N}(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin(t) & \cos(t) & 0 \\ -\cos(t) & -\sin(t) & 0 \end{vmatrix} \\
&= 0\mathbf{i} + 0\mathbf{j} + (-\sin(t)(-\sin(t)) - (-\cos(t))\cos(t))\mathbf{k} \\
&= \mathbf{k} = \langle 0, 0, 1\rangle
\end{aligned}$$

(c) Find the curvature $\kappa(t)$ of this curve.

We have many formulas for curvature, but given all of our previous computations, the best one is:

$$\kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{1}{5}$$

8. (12 points): You are standing in a desert whose temperature (in degrees fahrenheit) is described by the function $f(x, y) = 105 + 15\sin(x + y)$. Your current location is $(0, 0)$. Note: $(1, 0)$ is 1 mile East of your location, $(-1, 0)$ is 1 mile West, $(0, 1)$ is 1 mile North, and $(0, -1)$ is 1 mile South.



(a) Use a directional derivative to measure the change in temperature if you start walking North.

If I am facing North, then I am facing in the $\mathbf{j} = \langle 0, 1\rangle$ direction. Let's compute the gradient of f: $\nabla f = \langle f_x, f_y\rangle = \langle 15\cos(x + y), 15\cos(x + y)\rangle$. Thus, $\nabla f(0, 0) = \langle 15\cos(0), 15\cos(0)\rangle = \langle 15, 15\rangle$.

Therefore,

$$D_{\mathbf{j}}f(0, 0) = (\nabla f(0, 0)) \cdot\langle 0, 1\rangle = \langle 15, 15\rangle\cdot\langle 0, 1\rangle = 15$$

This means that if I start walking North from my current position, I should expect the temperature to increase by about 15°F per mile travelled.

(b) Walking in what direction will give a maximal *decrease* in temperature?

Remember that the gradient direction will give the maximal increase and the negative gradient gives the direction of maximal decrease. Thus I want to travel in the negative gradient direction.

Recall that $\nabla f(0, 0) = \langle 15, 15\rangle$. Thus the direction associated with $-\nabla f(0, 0) = \langle -15, -15\rangle$ will give the maximal decrease in temperature. This is the direction "**South-West**".

9. (10 points): Find all of the second partials of $z = x^2\ln(y)$.

The first partials are:

$$\frac{\partial z}{\partial x} = 2x\ln(y) \qquad \text{and} \qquad \frac{\partial z}{\partial y} = \frac{x^2}{y}$$

The second partials are:

$$\frac{\partial^2 z}{\partial x^2} = 2\ln(y), \qquad \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 z}{\partial y \partial x} = \frac{2x}{y}, \qquad \text{and} \qquad \frac{\partial^2 z}{\partial y^2} = -\frac{x^2}{y^2}$$