Math 251 Spring 2007
Exam #2 Answer Key

1. (12 points): Consider the function $f(x, y) = x^3 + x^2y - y^2 - 4y$. Find all of the critical points of $f(x, y)$ and classify them (i.e. as a minimum, maximum, saddle point, or nothing).

A point (x, y) is a critical point of $f(x, y)$ if $f_x(x, y) = f_y(x, y) = 0$. $f_x(x, y) = 3x^2 + 2xy$ and $f_y(x, y) = x^2 - 2y - 4$. Notice that $3x^2 + 2xy = x(3x + 2y) = 0$ only if $x = 0$ or $3x + 2y = 0$.

Consider the case $x = 0$. We also need $f_y(x, y) = 0$. So we need $0^2 - 2y - 4 = 0$ that is $-2y = 4$. So $y = -2$.

Our other case is that of $3x + 2y = 0$ that is $y = -(3/2)x$. We also need $f_y(x, y) = 0$. So we need $x^2 - 2\left(-\frac{3}{2}\right)x - 4 = 0$ that is $x^2 + 3x - 4 = 0$. Thus $(x + 4)(x - 1) = 0$ so that $x = -4$ or $x = 1$. If $x = -4$, then $y = -(3/2)(-4) = 6$. If $x = 1$, then $y = -3/2$.

We have found that $f(x, y)$ has three critical points: $(0, -2)$, $(-4, 6)$, and $(1, -3/2)$. To classify these points we compute the discriminant function $D(x, y) = f_{xx}(x, y)f_{yy}(x, y) - (f_{xy}(x, y))^2$.

Notice $f_{xx}(x, y) = 6x + 2y$, $f_{xy}(x, y) = 2x$, and $f_{yy}(x, y) = -2$. So that $D(x, y) = (6x + 2y)(-2) - (2x)^2 = -4x^2 - 12x - 4y$.

- $D(0, -2) = -4(0^2) - 12(0) - 4(-2) = 8 > 0$ and $f_{xx}(0, -2) = 6(0) + 2(-2) = -4 < 0$
- $D(-4, 6) = -4(-4)^2 - 12(-4) - 4(6) = -64 + 48 - 24 = -40 < 0$
- $D(1, -3/2) = -4(1^2) - 12(1) - 4(-3/2) = -4 - 12 + 6 = -10 < 0$

Answer: The critical points of $f(x, y)$ are: $(0, -2)$ – a local maximum, $(-4, 6)$ – a saddle point, and $(1, -3/2)$ – a saddle point.

2. (12 points): Let C be the line segment from $(1, 2, 3)$ to $(-1, 0, 1)$. Evaluate $\int_C z^2 - x^2\, ds$.

We parametrize the line segment as follows: $\mathbf{r}(t) = \langle 1, 2, 3\rangle(1-t) + \langle -1, 0, 1\rangle t$ for $0 \le t \le 1$. That is $\mathbf{r}(t) = \langle 1 - 2t, 2 - 2t, 3 - 2t\rangle$ for $0 \le t \le 1$.

We need to find ds. So we compute $\mathbf{r}'(t) = \langle -2, -2, -2\rangle$.
Thus $|\mathbf{r}'(t)| = \sqrt{(-2)^2 + (-2)^2 + (-2)^2} = 2\sqrt{3}$. Therefore, $ds = |\mathbf{r}'(t)|\, dt = 2\sqrt{3}\, dt$.

$$
\begin{aligned}
\int_C z^2 - x^2\, ds &= \int_0^1 \left((3 - 2t)^2 - (1 - 2t)^2\right) 2\sqrt{3}\, dt \\
&= \int_0^1 \left((4t^2 - 12t + 9) - (4t^2 - 4t + 1)\right) 2\sqrt{3}\, dt \\
&= \int_0^1 (-8t + 8)\, 2\sqrt{3}\, dt \\
&= 2\sqrt{3}\left(-4t^2 + 8t\right)\Big|_0^1 \\
&= 2\sqrt{3}\left((-4(1^2) + 8(1)) - (-4(0^2) + 8(0))\right)
\end{aligned}
$$

Answer: $\int_C z^2 - x^2\, ds = 8\sqrt{3}$

3. (14 points): Fix two positive real numbers a and b. Let $x = ar\cos(\theta)$ and $y = br\sin(\theta)$.

(a) Compute $\dfrac{\partial(x,y)}{\partial(r,\theta)}$.

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \det\left(\begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} \end{bmatrix}\right) = \det\left(\begin{bmatrix} a\cos(\theta) & b\sin(\theta) \\ -ar\sin(\theta) & br\cos(\theta) \end{bmatrix}\right) = abr\cos^2(\theta) + abr\sin^2(\theta)$$

Answer: abr

(b) Let $R = \left\{(x,y) \mid \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} \le 1\right\}$. Find the area of R by evaluating a double integral. *Hint:* Use the modified polar coordinates defined above.

R is the region bounded by the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

Notice that $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = \dfrac{a^2r^2\cos^2(\theta)}{a^2} + \dfrac{b^2r^2\sin^2(\theta)}{b^2} = r^2\cos^2(\theta) + r^2\sin^2(\theta) = r^2$. So we should have $0 \le r^2 = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} \le 1$. Since we want to include all of the interior of the ellipse, we should let $0 \le \theta \le 2\pi$. Therefore, we have the following:

$$\text{Area of } R = \iint_R 1\,dA = \int_0^{2\pi}\int_0^1 abr\,dr\,d\theta = \int_0^{2\pi} \frac{ab}{2}r^2\Big|_0^1\,d\theta = \int_0^{2\pi}\frac{ab}{2}\,d\theta = \frac{ab}{2}\theta\Big|_0^{2\pi} = \frac{ab}{2}2\pi$$

Answer: πab (Notice the special case $a = b = r$ gives πr^2.)

4. (13 points): Let E be the region inside both the cylinder $x^2 + y^2 = 1$ and the sphere $x^2 + y^2 + z^2 = 4$. Evaluate $\displaystyle\iiint_E 2(x^2+y^2)z\,dV$.

We have $z = \pm\sqrt{4 - x^2 - y^2}$ (this gives the "top" and "bottom" of our region E). Next x and y are bounded by $x^2 + y^2 = 1$. Thus we get $y = \pm\sqrt{1-x^2}$ and $-1 \le x \le 1$. Therefore,

$$\begin{aligned}
&\iiint_E 2(x^2+y^2)z\,dV \\
&= \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{-\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}} 2(x^2+y^2)z\,dz\,dy\,dx \\
&= \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (x^2+y^2)z^2\Big|_{-\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}}\,dy\,dx \\
&= \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (x^2+y^2)\left(\sqrt{4-x^2-y^2}\right)^2 - (x^2+y^2)\left(-\sqrt{4-x^2-y^2}\right)^2\,dy\,dx \\
&= \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 0\,dy\,dx
\end{aligned}$$

Answer: 0

Note: I originally wrote problem #4 to be a "triple integral in cylidrical coordinates" problem. However, when changing it around, I made a mistake that makes the answer come out to zero. Let's see how this integral transforms to cylindrical coordinates, notice that $x^2 + y^2 = 1$ changes to $r^2 = 1$. So the last two limits change to $0 \leq \theta \leq 2\pi$ and $0 \leq r \leq 1$. The limits for z change to $-\sqrt{4-r^2} \leq z \leq \sqrt{4-r^2}$. So we get the following integral (remember the Jacobian):

$$\iiint_E 2(x^2+y^2)z\,dV = \int_0^{2\pi}\int_0^1\int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} 2r^2 zr\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^1\int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} 2r^3 z\,dz\,dr\,d\theta = ... = 0$$

5. (12 points): Use Lagrange multipliers to find the maximum and minimum value of $f(x, y, z) = xyz$ subject to the constraint $x^2 + y^2 + z^2 = 3$.

To find the critical points, we need to solve the following equations:

$$\begin{aligned} \nabla f(x,y,z) &= \lambda \nabla g(x,y,z) \\ g(x,y,z) &= 3 \end{aligned}$$

where $f(x, y, z) = xyz$ is our objective function and $g(x, y, z) = x^2 + y^2 + z^2$ is our constraint function. Notice that $\nabla f(x, y, z) = \langle yz, xz, xy\rangle$ and $\nabla g(x, y, z) = \langle 2x, 2y, 2z\rangle$. So we have the following equations:

$$\begin{aligned} yz &= \lambda 2x \\ xz &= \lambda 2y \\ xy &= \lambda 2z \\ 3 &= x^2 + y^2 + z^2 \end{aligned}$$

If we multiply the first equation by x, the second by y, and the third by z, we get:

$$xyz = \lambda 2x^2 = \lambda 2y^2 = \lambda 2z^2$$

Now remember that $\lambda \neq 0$, so after dividing through by 2λ, we find that:

$$\frac{xyz}{2\lambda} = x^2 = y^2 = z^2$$

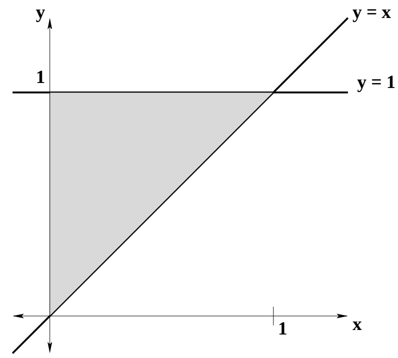
This means that $3 = x^2 + y^2 + z^2 = 3x^2$. Thus $x^2 = 1$ so that $x = \pm 1$ and because $x^2 = y^2 = z^2$ we must have that $y = \pm 1$ and $z = \pm 1$ also.

This gives us 8 ($= 2^3$) critical points: $(1, 1, 1)$, $(1, 1, -1)$, $(1, -1, 1)$, $(1, -1, -1)$, $(-1, 1, 1)$, $(-1, 1, -1)$, $(-1, -1, 1)$, and $(-1, -1, -1)$. But notice that $f(\pm 1, \pm 1, \pm 1) = \pm 1$.

Answer: The maximum value of $f(x, y, z)$ subject to the constraint $x^2 + y^2 + z^2 = 3$ is 1 and the minimum value is -1.

6. (11 points): Evaluate $\displaystyle\int_0^1\int_x^1 e^{y^2}\,dy\,dx$.

Since we cannot integrate the expression "e^{y^2}", we will reverse the order of integration to see if that helps. First, let's sketch a graph of the region of integration.



The region of integration: $R = \{(x,y) \in \mathbb{R}^2 \mid x \le y \le 1,\ 0 \le x \le 1\}$.

Notice that $R = \{(x,y) \in \mathbb{R}^2 \mid 0 \le x \le y,\ 0 \le y \le 1\}$ when described as a "Type II" region. So we get that:

$$\int_0^1 \int_x^1 e^{y^2}\, dy\, dx = \int_0^1 \int_0^y e^{y^2}\, dx\, dy = \int_0^1 x e^{y^2}\Big|_0^y\, dy = \int_0^1 y e^{y^2}\, dy = \frac{1}{2} e^{y^2}\Big|_0^1$$

Answer: $\frac{1}{2}(e-1)$

7. (13 points): Consider the vector field $\mathbf{F}(x,y) = \left(2xe^{2y} + 3x^2\right)\mathbf{i} + \left(2x^2e^{2y} + \cos(y)\right)\mathbf{j}$.

(a) Show that $\mathbf{F}(x,y)$ is **conservative** by finding a potential function.

Consider the following integrals:

$$\int 2xe^{2y} + 3x^2\, dx = x^2e^{2y} + x^3 + g_1(y)$$

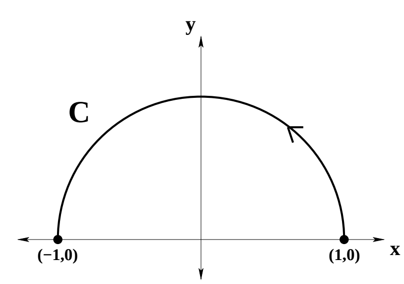
$$\int 2x^2e^{2y} + \cos(y)\, dy = x^2e^{2y} + \sin(y) + g_2(x)$$

Combining the above results gives us our potential function.

Answer: $f(x,y) = x^2e^{2y} + x^3 + \sin(y) + C$ (where C is some constant)

(b) Let C be the upper-half of the circle $x^2 + y^2 = 1$ oriented counter-clockwise. Evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$.

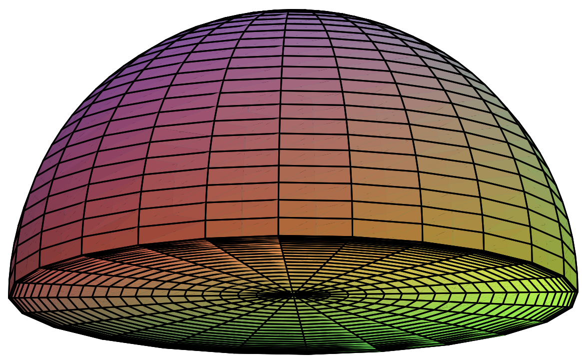
We all ready showed that $\mathbf{F}(x,y)$ is conservative. So instead of computing the line integral directly, we can use the fundamental theorem of line integrals. From part (a) we have the potential function $f(x,y) = x^2e^{2y} + x^3 + \sin(y)$. Next notice that our curve C begins at $(1,0)$ and ends at $(-1,0)$.



$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C \nabla f\cdot d\mathbf{r} = f(-1,0) - f(1,0) = \left((-1)^2e^0 + (-1)^3 + \sin(0)\right) - \left(1^2e^0 + 1^3 + \sin(0)\right)$$

Answer: -2.

8. (13 points): Find the centroid of the region, E, inside the sphere $x^2+y^2+z^2=4$ and above the xy-plane. *Hint:* The volume of half of a sphere or radius r is $\frac{2}{3}\pi r^3$. Use this to find m. Also, $\bar{x}=\bar{y}=0$ by symmetry.



The region E.

We need to compute M_{xy} and m to find $\bar{z}$, but we know that $m = \text{Volume of } E = \frac{2}{3}\pi 2^3 = \frac{16}{3}\pi$ since E is half of a sphere of radius $r=2$. So we must compute: $M_{xy} = \iiint_E z\,dV$. Since E is half of a sphere, it makes sense to compute this triple integral in spherical coordinates. We know that $z=\rho\cos(\phi)$ and the Jacobian is $\rho^2\sin(\phi)$, and in spherical coordinates E is described by $0\le\rho\le 2$, $0\le\theta\le 2\pi$, and $0\le\phi\le\frac{\pi}{2}$ (ϕ ranges to $\pi/2$ instead of π because E is only the upper-half of the sphere). Therefore:

$$
\begin{aligned}
M_{xy} &= \iiint_E z\,dV \\
&= \int_0^{\frac{\pi}{2}}\int_0^{2\pi}\int_0^2 \rho\cos(\phi)\rho^2\sin(\phi)\,d\rho\,d\theta\,d\phi \\
&= \int_0^{\frac{\pi}{2}}\sin(\phi)\cos(\phi)\,d\phi\int_0^{2\pi}d\theta\int_0^2\rho^3\,d\rho \\
&= \left(\frac{1}{2}\sin^2(\phi)\Big|_0^{\frac{\pi}{2}}\right)\cdot(2\pi)\cdot\left(\frac{1}{4}\rho^4\Big|_0^2\right) \\
&= \frac{1}{2}\left(1^2-0^2\right)\cdot(2\pi)\cdot\frac{1}{4}\left(2^4-0^4\right) \\
&= \frac{1}{2}(2\pi)\frac{16}{4} = 4\pi
\end{aligned}
$$

[Note: We can "factor" the integral (as in the third equality) since the limits are all constants.] Therefore, $\bar{z} = \frac{M_{xy}}{m} = \frac{4\pi}{\frac{16}{3}\pi} = \frac{3}{4}$.

Answer: The centroid of E is $(\bar{x},\bar{y},\bar{z}) = \left(0,0,\frac{3}{4}\right)$.