SAMPLE FINAL EXAM - Answer Key

1. (9 points): Lines and Planes

(a) Find parametric equations for the line which passes through $(1, 2, 3)$ and is parallel to the vector $\langle 1, 0, 1\rangle$.

A vector equation for this line is $\mathbf{r}(t) = \langle 1, 2, 3\rangle + \langle 1, 0, 1\rangle t$.

The corresponding parametric equations are: $\begin{array}{rcl} x(t) & = & 1+t \\ y(t) & = & 2 \\ z(t) & = & 3+t \end{array}$.

(b) Find parametric equations for the line which passes through $(4, 5, 6)$ and is parallel to the vector $\langle 0, 1, 2\rangle$.

A vector equation for this line is $\mathbf{r}(t) = \langle 4, 5, 6\rangle + \langle 0, 1, 2\rangle t$.

The corresponding parametric equations are: $\begin{array}{rcl} x(t) & = & 4 \\ y(t) & = & 5+t \\ z(t) & = & 6+2t \end{array}$.

(c) Find the equation of the plane which is parallel to the lines from parts (a) and (b) and passes through the point $(-1, 0, 1)$.

To be parallel to both lines the plane must be parallel to both of their direction vectors. So to find a normal vector for the plane, we can simply compute the cross product of these two direction vectors.

$$\begin{aligned} \langle 1,0,1\rangle \times \langle 0,1,2\rangle &= \det\left(\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}\right) \\ &= (0(2)-1(1))\,\mathbf{i} - (1(2)-0(1))\,\mathbf{j} + (1(1)-0(0))\,\mathbf{k} \\ &= \langle -1,-2,1\rangle \end{aligned}$$

We have the following equation for the plane: $-1(x-(-1)) - 2(y-0) + 1(z-1) = 0$ which is $-x - 2y + z - 2 = 0$.

2. (9 points): Consider the curve $\mathbf{r}(t) = \langle \sin(t), t, \cos(t)\rangle$.

(a) Find the curvature of $\mathbf{r}(t)$.

Since we need to compute $\mathbf{T}(t)$ for part (b), let use the formula: $\kappa(t) = \dfrac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}$.

$\mathbf{r}'(t) = \langle \cos(t), 1, -\sin(t)\rangle$ so that $|\mathbf{r}'(t)| = \sqrt{\cos^2(t) + 1^2 + (-\sin(t))^2} = \sqrt{2}$. Therefore, $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \dfrac{1}{\sqrt{2}}\langle \cos(t), 1, -\sin(t)\rangle$. Next, we have that $\mathbf{T}'(t) = \dfrac{1}{\sqrt{2}}\langle -\sin(t), 0, -\cos(t)\rangle$. Thus $|\mathbf{T}'(t)| = \dfrac{1}{\sqrt{2}}\sqrt{(-\sin(t))^2 + (-\cos(t))^2} = \dfrac{1}{\sqrt{2}}$.

So we have found that $\kappa(t) = \dfrac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \dfrac{\frac{1}{\sqrt{2}}}{\sqrt{2}} = \dfrac{1}{2}$.

(b) Find $\mathbf{T}(\pi)$, $\mathbf{N}(\pi)$, and $\mathbf{B}(\pi)$.

Using part (a), we get that $\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \dfrac{1}{\frac{1}{\sqrt{2}}}\dfrac{1}{\sqrt{2}}\langle -\sin(t), 0, -\cos(t)\rangle$
$= \langle -\sin(t), 0, -\cos(t)\rangle$.

Plugging in $t = \pi$, we get that $\mathbf{T}(\pi) = \dfrac{1}{\sqrt{2}}\langle -1, 1, 0\rangle$ and $\mathbf{N}(\pi) = \langle 0, 0, 1\rangle$.

$$\begin{aligned}
\mathbf{B}(\pi) &= \mathbf{T}(\pi) \times \mathbf{N}(\pi) \\
&= \det\left(\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) \\
&= \left(\frac{1}{\sqrt{2}}(1) - 0(0)\right)\mathbf{i} - \left(-\frac{1}{\sqrt{2}}(1) - 0(0)\right)\mathbf{j} + \left(-\frac{1}{\sqrt{2}}(0) - 0\frac{1}{\sqrt{2}}\right)\mathbf{k} \\
&= \frac{1}{\sqrt{2}}\langle 1, 1, 0\rangle
\end{aligned}$$

3. (7 points): Let $e^{xy+z} - xy - z = 0$. Find $\dfrac{\partial z}{\partial x}$.

We have the level surface $F(x, y, z) = e^{xy+z} - xy - z = 0$. Notice that $F_x = ye^{xy+z} - y$ and $F_z = e^{xy+z} - 1$.

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{ye^{xy+z} - y}{e^{xy+z} - 1} = -y$$

4. (9 points): Find the minimum and maximum value of $f(x, y) = 4 - x^2 - y^2$ subject to the constraint $x^2 + 2y^2 \leq 1$.

First, we need to find all of the critical points for which $x^2+2y^2 < 1$. Notice that $f_x = -2x$ and $f_y = -2y$. Therefore, $f_x = 0$ and $f_y = 0$ only when $(x, y) = (0, 0)$ and $0^2+2(0^2) = 0 < 1$.

[*Note:* We don't need to classify this point as a local min, local max, or saddle point since we are looking for an absolute max and min. However, if we did check we would find that $D(0, 0) = f_{xx}(0, 0)f_{yy}(0, 0) - (f_{xy}(0, 0))^2 = (-2)(-2) - 0^2 = 4 > 0$ and $f_{xx}(0, 0) = -2 < 0$. Also, $f(0, 0) = 4 - 0^2 - 0^2 = 4$ so that $(0, 0, 4)$ is a local maximum.]

Next, we need to find all of the critical points on the boundary of this region – that is where $x^2 + 2y^2 = 1$. To do this we will use the method of Lagrange multipliers.

Let $g(x, y) = x^2 + 2y^2$ so that $g(x, y) = 1$ is our constraint. Then $\nabla f = \langle -2x, -2y\rangle$ and $\nabla g = \langle 2x, 4y\rangle$. The vector equation $\nabla f = \lambda \nabla g$ along with the constraint $g(x, y) = 1$ give us the following equations: $-2x = \lambda 2x$, $-2y = \lambda 4y$ and $x^2 + 2y^2 = 1$.

If $x \neq 0$ we can cancel x off of the first equation and find that $-2 = \lambda 2$ so that $\lambda = -1$. In this case, $-2y = -4y$ so that $y = 0$. Now if $y = 0$, we have that $x^2 + 0^2 = 1$ so that $x = \pm 1$.

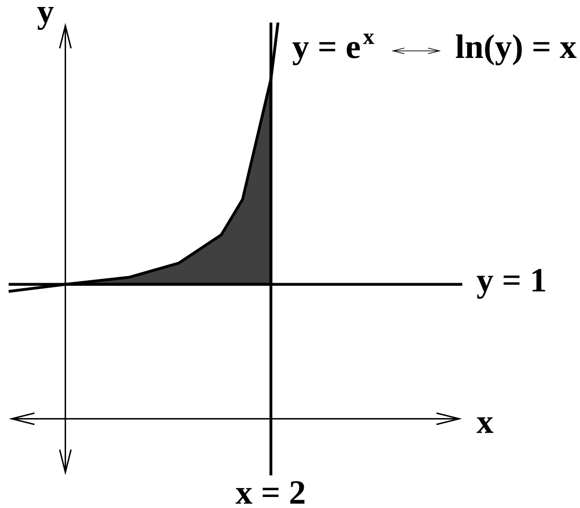
On the other hand if $x = 0$, then $0^2 + 2y^2 = 1$ so that $y = \pm\dfrac{1}{\sqrt{2}}$.

To sum up we have the following critical points: $(0,0)$, $(\pm 1, 0)$, and $\left(0, \pm\frac{1}{\sqrt{2}}\right)$. Plugging these points into $f(x,y)$, we get 4, 3, and 7/2.

Therefore, $f(x,y)$ constrained to $x^2 + 2y^2 \leq 1$ has the absolute minimum value of 3 when $(x,y) = (\pm 1, 0)$ and the absolute maximum value of 4 when $(x,y) = (0,0)$.

5. (7 points): Rewrite the following integral with the order of integration reversed:

$$\int_0^2 \int_1^{e^x} f(x,y)\, dy\, dx$$



The region of integration.

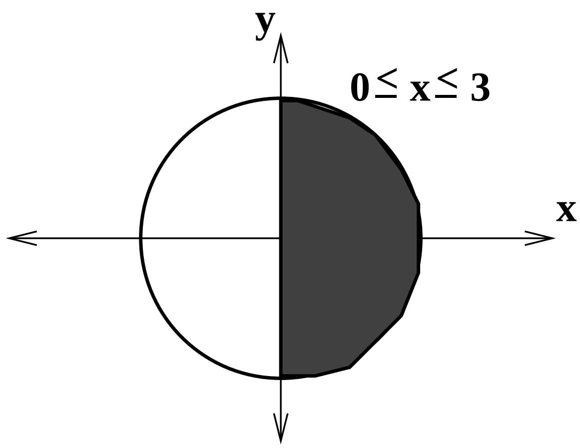
$$\int_0^2 \int_1^{e^x} f(x,y)\, dy\, dx = \int_1^{e^2} \int_{\ln(y)}^{2} f(x,y)\, dx\, dy$$

6. (8 points): Evaluate the following integral:

$$\int_0^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_{-\sqrt{9-x^2-y^2}}^{\sqrt{9-x^2-y^2}} \frac{1}{\sqrt{x^2+y^2+z^2}}\, dz\, dy\, dx$$

First, notice the term $\frac{1}{\sqrt{x^2+y^2+z^2}}$. Recall that $\rho^2 = x^2 + y^2 + z^2$. Also, notice that the limits of integration tell us that we are integrating over part of a sphere. So it makes sense to switch to spherical coordinates.

The "z" bounds go from $z = -\sqrt{9-x^2-y^2}$ to $z = \sqrt{9-x^2-y^2}$. These are the equations for the bottom and the top of a sphere centered at the origin of radius 3. The "y" bounds go from $y = -\sqrt{9-x^2}$ to $y = \sqrt{9-x^2}$. These are the equations for the bottom-half and top-half of a circle centered at the origin of radius 3. So far it looks like we're integrating over the whole sphere. However, notice that the "x" bounds go from $x = 0$ to $x = 3$ (not $x = -3$ to $x = 3$). So in fact we are integrating over half of the sphere. Just choosing positive x's corresponds to restricting the angle θ so that $-\pi/2 \leq \theta \leq \pi/2$.



The xy-Region of Integration.

$$\begin{aligned}
\int_0^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_{-\sqrt{9-x^2-y^2}}^{\sqrt{9-x^2-y^2}} \frac{1}{\sqrt{x^2+y^2+z^2}}\,dz\,dy\,dx &= \int_0^{\pi}\int_{-\pi/2}^{\pi/2}\int_0^3 \frac{1}{\sqrt{\rho^2}}\rho^2\sin(\phi)\,d\rho\,d\theta\,d\phi \\
&= \int_0^{\pi}\int_{-\pi/2}^{\pi/2}\int_0^3 \rho\sin(\phi)\,d\rho\,d\theta\,d\phi \\
&= \left(\int_0^{\pi}\sin(\phi)\,d\phi\right)\left(\int_{-\pi/2}^{\pi/2} d\theta\right)\left(\int_0^3 \rho\,d\rho\right) \\
&= \left(-\cos(\phi)|_0^{\pi}\right)(\pi)\left(\frac{1}{2}\rho^2\Big|_0^3\right) \\
&= (2)(\pi)\left(\frac{1}{2}3^2\right) = 9\pi
\end{aligned}$$

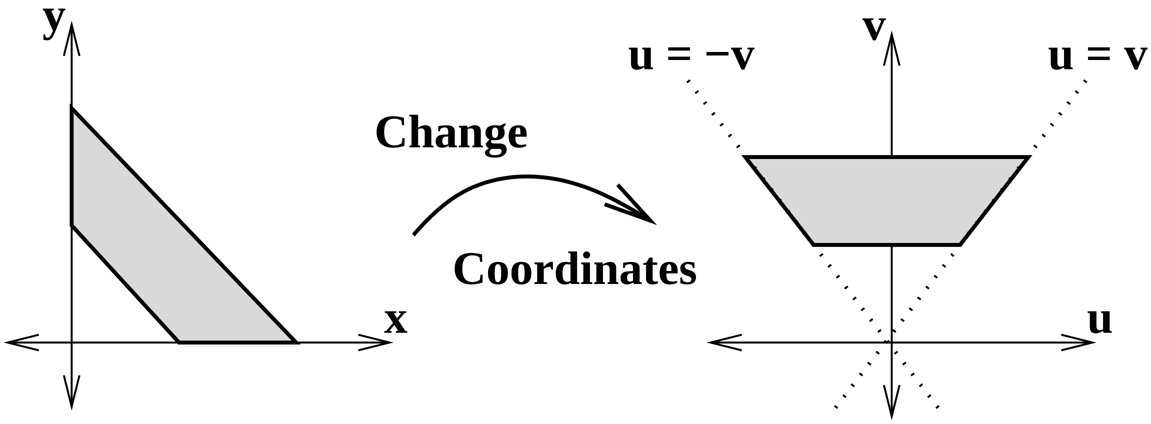
Note: Because of the spherical symmetry of the function $f(x,y,z) = \dfrac{1}{\sqrt{x^2+y^2+z^2}}$ you could integrate over any half of the sphere and get the same answer. In fact, you could integrate over the whole sphere and divide your answer by 2. But be warned...this only works because of the symmetry of f!

7. (9 points): Evaluate the following integral where R is the trapezoidal region with vertices $(1,0)$, $(2,0)$, $(0,2)$, and $(0,1)$:

$$\iint_R \cos\left(\frac{y-x}{y+x}\right)\,dA$$

Hint: Choose a change of variables that makes $\frac{y-x}{y+x}$ simple.

Let's set $u = y - x$ and $v = y + x$ and see what happens to our region of integration. The point $(x,y) = (1,0)$ becomes $(u,v) = (-1,1)$; $(2,0)$ becomes $(-2,2)$; $(0,2)$ becomes $(2,2)$; and $(0,1)$ becomes $(1,1)$. This means that $1 \le v \le 2$ and $-v \le u \le v$. So our bounds have simplified along with our function $\cos\left(\dfrac{y-x}{y+x}\right) = \cos\left(\dfrac{u}{v}\right)$.



Changing Coordinates

We need to compute the Jacobian of this coordinate transformation. In order to do so we need to solve our uv-equations in terms of x and y. Notice that $u+v = (y-x)+(y+x) = 2y$ and that $-u+v = -(y-x)+(y+x) = 2x$. Thus $x = -\frac{1}{2}u + \frac{1}{2}v$ and $y = \frac{1}{2}u + \frac{1}{2}v$. So we get the following Jacobian:

$$\frac{\partial(x,y)}{\partial(u,v)} = \det\left(\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}\right) = \det\left(\begin{bmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}\right) = -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2}$$

$$\begin{aligned}
\iint_R \cos\left(\frac{y-x}{y+x}\right)\,dA &= \int_1^2 \int_{-v}^{v} \cos\left(\frac{u}{v}\right)\left|-\frac{1}{2}\right|\,du\,dv \\
&= \int_1^2 \frac{1}{2}v\sin\left(\frac{u}{v}\right)\Big|_{-v}^{v}\,dv \\
&= \int_1^2 \frac{1}{2}v\sin\left(\frac{v}{v}\right) - \frac{1}{2}v\sin\left(\frac{-v}{v}\right)\,dv \\
&= \int_1^2 v\sin(1)\,dv \\
&= \frac{\sin(1)}{2}v^2\Big|_1^2 \\
&= \frac{\sin(1)}{2}(2^2-1^2) = \frac{3}{2}\sin(1)
\end{aligned}$$

Note: $\sin\left(\frac{-v}{v}\right) = \sin(-1) = -\sin(1)$ since $\sin(x)$ is a odd function – that is $\sin(-x) = -\sin(x)$ for all x.

8. (8 points): Consider the following vector field:

$$\mathbf{F}(x, y, z) = (yz + 2x)\mathbf{i} + (xz + z)\mathbf{j} + (xy + y)\mathbf{k}$$

(a) Show that $\mathbf{F}$ is conservative by finding a potential function.

$$\int yz + 2x\, dx = xyz + x^2 + g_1(y, z)$$
$$\int xz + z\, dy = xyz + yz + g_2(x, z)$$
$$\int xy + y\, dz = xyz + yz + g_3(x, y)$$

Therefore, $f(x, y, z) = xyz + x^2 + yz + C$ (where C is an arbitrary constant) is a potential function for $\mathbf{F}(x, y, z)$ (that is $\nabla f = \mathbf{F}$).

(b) Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ where C is the curve given by $\mathbf{r}(t) = \langle e^t, t, te^t \rangle$ where $0 \leq t \leq 1$ and

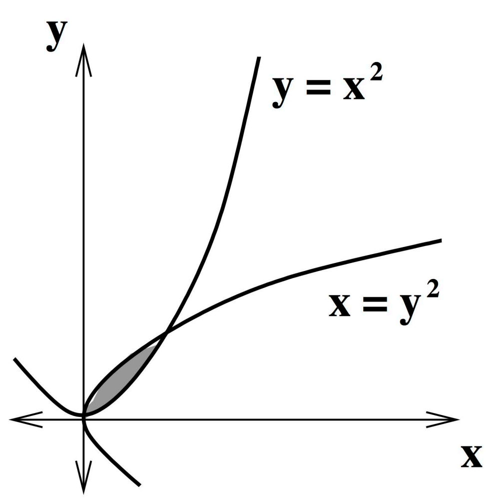
Since $\mathbf{F}$ is conservative, we can use the fundamental theorem of line integrals. Notice that C starts at the point $\mathbf{r}(0) = \langle e^0, 0, 0e^0 \rangle = \langle 1, 0, 0 \rangle$ and ends at the point $\mathbf{r}(1) = \langle e^1, 1, 1e^1 \rangle = \langle e, 1, e \rangle$.

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} &= f(\mathbf{r}(1)) - f(\mathbf{r}(0)) = f(e, 1, e) - f(1, 0, 0) \\ &= \big(e(1)e + e^2 + 1(e)\big) - \big(1(0)0 + 1^2 + 0(0)\big) = 2e^2 + e - 1 \end{aligned}$$

9. (8 points): Use Green's Theorem to evaluate the line integral along the given positively oriented curve.

$$\int_C (y + e^{\sqrt{x}})dx + (2x + \cos(y^2))dy$$

where C is the boundary of the region enclosed by the parabolas $y = x^2$ and $x = y^2$.



The Region Bounded by the Curve C.

Noticing that $Q(x,y) = 2x + \cos(y^2)$ and $P(x,y) = y + e^{\sqrt{x}}$, we apply Green's Theorem and get...

$$\begin{aligned}
\int_C (y + e^{\sqrt{x}})dx + (2x + \cos(y^2))dy &= \int_0^1 \int_{x^2}^{\sqrt{x}} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dy\, dx = \int_0^1 \int_{x^2}^{\sqrt{x}} 2 - 1\, dy\, dx \\
&= \int_0^1 y\Big|_{x^2}^{\sqrt{x}}\, dx = \int_0^1 \sqrt{x} - x^2\, dx \\
&= \left.\frac{x^{3/2}}{3/2} - \frac{x^3}{3}\right|_0^1 = \frac{2}{3}(1^{3/2}) - \frac{1}{3}(1^3) = \frac{1}{3}
\end{aligned}$$

10. (8 points): Let S be the surface given by $x^2 + y^2 = 9$ and $1 \leq z \leq 4$.

(a) Find a parametrization of the surface.

There are infinitely many ways to parametrize this surface. But one seems to be most natural. Notice that we are dealing with a piece of a circular cylinder. This suggests cylindrical coordinates might be helpful. Indeed, notice that the cylinder has radius 3 (because $x^2 + y^2 = r^2 = 9$). This leads us to the following parametrization:

$$\mathbf{r}(\theta, z) = \langle 3\cos(\theta), 3\sin(\theta), z\rangle$$

where $0 \leq \theta \leq 2\pi$ and $1 \leq z \leq 4$.

Note: If we tried to force rectangular coordinates, we would get $\mathbf{r}_1(x,z) = \langle x, \sqrt{9 - x^2}, z\rangle$ and also $\mathbf{r}_2(x,z) = \langle x, -\sqrt{9 - x^2}, z\rangle$ where for both $\mathbf{r}_1(x,z)$ and $\mathbf{r}_2(x,z)$ we restrict $-3 \leq x \leq 3$ and $1 \leq z \leq 4$. If we try to parametrize our surface this way, we get two pieces. Our original choice is clearly much better (and more natural).

(b) Find an orientation for S.

We can do this two ways. First, let's use our parametrization found in part (a). $\mathbf{r}_\theta = \langle -3\sin(\theta), 3\cos(\theta), 0\rangle$ and $\mathbf{r}_z = \langle 0, 0, 1\rangle$. We then compute their cross product and get...

$$\begin{aligned}
\mathbf{r}_\theta \times \mathbf{r}_z &= \det\left(\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3\sin(\theta) & 3\cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) \\
&= (3\cos(\theta)(1) - 0(0))\,\mathbf{i} - (-3\sin(\theta)(1) - 0(0))\,\mathbf{j} + (-3\sin(\theta)(0) - 3\cos(\theta)(0))\,\mathbf{k} \\
&= \langle 3\cos(\theta), 3\sin(\theta), 0\rangle
\end{aligned}$$

$|\mathbf{r}_\theta \times \mathbf{r}_z| = \sqrt{9\cos^2(\theta) + 9\sin^2(\theta) + 0^2} = \sqrt{9} = 3$ and thus we get the two orientations:

$$\mathbf{n} = \frac{\mathbf{r}_\theta \times \mathbf{r}_z}{|\mathbf{r}_\theta \times \mathbf{r}_z|} = \langle \cos(\theta), \sin(\theta), 0\rangle \qquad \text{and} \qquad -\mathbf{n} = \langle -\cos(\theta), -\sin(\theta), 0\rangle$$

The first orientation $\mathbf{n}$ points "out" from the cylinder and the second $-\mathbf{n}$ points "inside" the cylinder.

Alternate Derivation: Notice that surface is a level surface $F(x,y,z) = x^2+y^2 = 9$. So we can get orientations as follows:

$$\mathbf{n} = \frac{\nabla F}{|\nabla F|} = \frac{\langle 2x, 2y, 0\rangle}{\sqrt{4x^2+4y^2}} = \frac{\langle 2x, 2y, 0\rangle}{\sqrt{4(9)}} = \left\langle \frac{x}{3}, \frac{y}{3}, 0\right\rangle$$

And also, $-\mathbf{n}$.

Notice if we plug our parametrization into $\mathbf{n}$ we get...

$$\mathbf{n} = \left\langle \frac{x}{3}, \frac{y}{3}, 0\right\rangle = \left\langle \frac{3\cos(\theta)}{3}, \frac{3\sin(\theta)}{3}, 0\right\rangle = \langle \cos(\theta), \sin(\theta), 0\rangle$$

The same as before. Personally, I think the second way is much easier!

(c) Find the equation of the tangent plane to S at the point $(3,0,2)$.

To find the equation of a plane we need a point – $(3,0,2)$ – and a normal vector – $\mathbf{n}(3,0,2) = \left\langle \frac{3}{3}, \frac{0}{3}, 0\right\rangle$ (remember that orientations are nothing more than formuals for unit normals of tangents planes).

So the tangent plane at $(3,0,2)$ is $1(x-3)+0(y-0)+0(z-2)=0$ which is $x=3$.

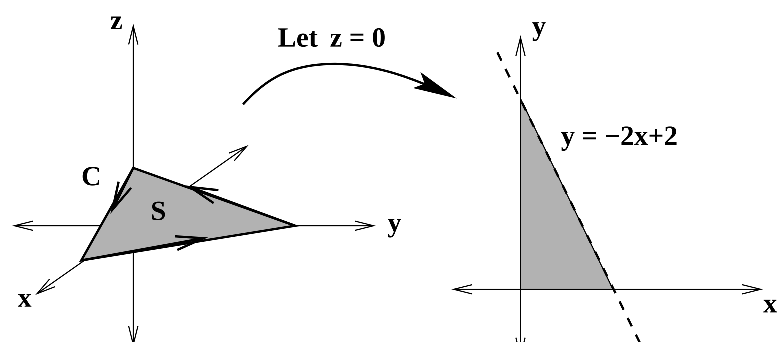
Note: I used the unparametrized form the the orientation to find the normal vector. If you wanted to use the parametrized form, you would first need to determine "where" the point $(3,0,2)$ is on your surface. Namely you would have to solve $\mathbf{r}(\theta,z) = \langle 3,0,2\rangle$. This would give you $3\cos(\theta)=3$, $3\sin(\theta)=0$, and $z=2$. Which means that $\theta=0$ and $z=2$. Plugging this into the "parametrized" version of the orientation from part (b), we get $\mathbf{n}(0,2) = \langle \cos(0), \sin(0), 0\rangle = \langle 1,0,0\rangle$. Which is exactly what we found before.

(d) Find the surface area of S.

$$\iint_S 1\,dS = \int_1^4 \int_0^{2\pi} |\mathbf{r}_\theta \times \mathbf{r}_z|\,d\theta\,dz = \int_1^4 \int_0^{2\pi} 3\,d\theta\,dz = \int_1^4 6\pi\,dz = 6\pi(4-1) = 18\pi$$

Alternatively, notice that S is just the side of a cylinder. So by basic highschool geometry we have: surface area = diameter of the circle × height $= 2\cdot 3\cdot \pi \times (4-1) = 18\pi$.

11. (9 points): Find $\int_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F}(x,y,z) = e^{-x}\,\mathbf{i} + e^x\,\mathbf{j} + e^z\,\mathbf{k}$ and C is the boundary of the part of the plane $2x+y+2z=2$ in the first octant ($x \geq 0$, $y \geq 0$, $z \geq 0$). Orient C to be counterclockwise when viewed from above. *Hint:* Stoke's Theorem.



The Region Bounded by the Curve C and Bounds for x and y.

To apply Stoke's Theorem we need to compute curl($\mathbf{F}$), parametrize our surface, and find an orientation.

$$\text{curl}(\mathbf{F}) = \det\left(\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^{-x} & e^{x} & e^{z} \end{bmatrix}\right) = (0-0)\,\mathbf{i} - (0-0)\,\mathbf{j} + (e^x - 0)\,\mathbf{k} = \langle 0, 0, e^x \rangle$$

We can parametrize our surface as follows: $\mathbf{r}(x,y) = \left\langle x, y, 1 - x - \dfrac{1}{2}y \right\rangle$ where $0 \le y \le -2x+2$ and $0 \le x \le 1$.

Notice that $\mathbf{r}_x = \langle 1, 0, -1 \rangle$ and $\mathbf{r}_y = \langle 0, 1, -1/2 \rangle$ so that

$$\begin{aligned} \mathbf{r}_x \times \mathbf{r}_y &= \det\left(\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -1 \\ 0 & 1 & -1/2 \end{bmatrix}\right) \\ &= (0(-1/2) - 1(-1))\,\mathbf{i} - (1(-1/2) - 0(-1))\,\mathbf{j} + (1(1) - 0(0))\,\mathbf{k} = \langle 1, 1/2, 1 \rangle \end{aligned}$$

Notice that this vector points in the positive xyz-direction which matches the orientation of our curve C (looking "down" on our surface from the "top", C is oriented "counter-clockwise").

Therefore, applying Stoke's Theorem, we get...

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^1 \int_0^{-2x+2} \text{curl}(\mathbf{F}) \cdot (\mathbf{r}_x \times \mathbf{r}_y)\, dy\, dx = \int_0^1 \int_0^{-2x+2} \langle 0, 0, e^x \rangle \cdot \langle 1, 1/2, 1 \rangle\, dy\, dx \\ &= \int_0^1 \int_0^{-2x+2} e^x\, dy\, dx = \int_0^1 (-2x+2)e^x\, dx \\ &= (-2x+4)e^x\Big|_0^1 = (-2+4)e^1 - 4e^0 = 2e - 4 \end{aligned}$$

12. (9 points): Find $\displaystyle\iint_S \mathbf{F} \cdot d\mathbf{S}$ where $\mathbf{F}(x,y,z) = x^3\,\mathbf{i} + 2xz^2\,\mathbf{j} + 3y^2z\,\mathbf{k}$, where S is the surface of the solid bounded by the paraboloid $z = 4 - x^2 - y^2$ and the xy-plane and S is positively oriented.

Since S is a postively oriented surface which is the boundary of a solid region, we can apply the Divergence Theorem. Let E be the solid bounded by S and notice that

$$\text{div}(\mathbf{F}) = \frac{\partial}{\partial x}\left(x^3\right) + \frac{\partial}{\partial y}\left(2xz^2\right) + \frac{\partial}{\partial z}\left(3y^2z\right) = 3x^2 + 3y^2.$$

Thus we get...

$$\begin{aligned} \iint_S \mathbf{F} \cdot d\mathbf{S} &= \iiint_E \text{div}(\mathbf{F})\, dz\, dy\, dx = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_0^{4-x^2-y^2} 3(x^2+y^2)\, dz\, dy\, dx \\ &= \int_0^{2\pi} \int_0^2 \int_0^{4-r^2} 3(r^2)r\, dz\, dr\, d\theta = \int_0^{2\pi} d\theta \int_0^2 3r^3 z\Big|_0^{4-r^2}\, dr \\ &= 2\pi \int_0^2 12r^3 - 3r^5\, dr = 2\pi\left(3r^4 - \frac{1}{2}r^6\Big|_0^2\right) \\ &= 2\pi\left(3(2^4) - 2^5\right) = 2\pi(3-2)2^4 = 32\pi \end{aligned}$$

Note: We switched to cylidrical coordinates to help make the integrals come out nice.