Name ______________________________ Math 251, Final Exam

SAMPLE FINAL EXAM

Be sure to show all your work. Unsupported answers will receive no credit.

A formula sheet is supplied, for your reference.

Use the backs of the exam pages for scratchwork or for continuation of your answers, if necessary.

Problem No.	**Pts Possible**	**Points**
1	9	
2	9	
3	7	
4	9	
5	7	
6	8	
7	9	
8	8	
9	8	
10	8	
11	9	
12	9	
Total	100	

1. (9 points): Lines and Planes

(a) Find parametric equations for the line which passes through $(1, 2, 3)$ and is parallel to the vector $\langle 1, 0, 1 \rangle$.

(b) Find parametric equations for the line which passes through $(4, 5, 6)$ and is parallel to the vector $\langle 0, 1, 2 \rangle$.

(c) Find the equation of the plane which is parallel to the lines from parts (a) and (b) and passes through the point $(-1, 0, 1)$.

2. (9 points): Consider the curve $\mathbf{r}(t) = \langle \sin(t), t, \cos(t) \rangle$.

(a) Find the curvature of $\mathbf{r}(t)$.

(b) Find $\mathbf{T}(\pi)$, $\mathbf{N}(\pi)$, and $\mathbf{B}(\pi)$.

3. (7 points): Let $e^{xy+z} - xy - z = 0$. Find $\dfrac{\partial z}{\partial x}$.

4. (9 points): Find the minimum and maximum value of $f(x, y) = 4 - x^2 - y^2$ subject to the constraint $x^2 + 2y^2 \leq 1$.

5. (7 points): Rewrite the following integral with the order of integration reversed:

$$\int_0^2 \int_1^{e^x} f(x,y)\,dy\,dx$$

6. (8 points): Evaluate the following integral:

$$\int_0^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_{-\sqrt{9-x^2-y^2}}^{\sqrt{9-x^2-y^2}} \frac{1}{\sqrt{x^2+y^2+z^2}}\, dz\, dy\, dx$$

7. (9 points): Evaluate the following integral where R is the trapezoidal region with vertices $(1,0)$, $(2,0)$, $(0,2)$, and $(0,1)$:

$$\iint_R \cos\left(\frac{y-x}{y+x}\right)\, dA$$

Hint: Choose a change of variables that makes $\frac{y-x}{y+x}$ simple.

8. (8 points): Consider the following vector field:

$$\mathbf{F}(x,y,z) = (yz+2x)\mathbf{i} + (xz+z)\mathbf{j} + (xy+y)\mathbf{k}$$

(a) Show that $\mathbf{F}$ is conservative by finding a potential function.

(b) Evaluate the line integral $\int_C \mathbf{F}\cdot d\mathbf{r}$ where C is the curve given by $\mathbf{r}(t) = \langle e^t, t, te^t\rangle$ where $0 \le t \le 1$ and

9. (8 points): Use Green's Theorem to evaluate the line integral along the given positively oriented curve.

$$\int_C (y + e^{\sqrt{x}})dx + (2x + \cos(y^2))dy$$

where C is the boundary of the region enclosed by the parabolas $y = x^2$ and $x = y^2$.

10. (8 points): Let S be the surface given by $x^2 + y^2 = 9$ and $1 \leq z \leq 4$.

(a) Find a parametrization of the surface.

(b) Find an orientation for S.

(c) Find the equation of the tangent plane to S at the point $(3, 0, 2)$.

(d) Find the surface area of S.

11. (9 points): Find $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = e^{-x}\,\mathbf{i} + e^{x}\,\mathbf{j} + e^{z}\,\mathbf{k}$ and C is the boundary of the part of the plane $2x + y + 2z = 2$ in the first octant ($x \geq 0$, $y \geq 0$, $z \geq 0$). Orient C to be counterclockwise when viewed from above. *Hint:* Stoke's Theorem.

12. (9 points): Find $\iint_S \mathbf{F}\cdot d\mathbf{S}$ where $\mathbf{F}(x, y, z) = x^3\,\mathbf{i} + 2xz^2\,\mathbf{j} + 3y^2z\,\mathbf{k}$, where S is the surface of the solid bounded by the paraboloid $z = 4 - x^2 - y^2$ and the xy-plane and S is positively oriented.

Formulas for the Final Exam

$$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$$
$$|\mathbf{a}\times\mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\theta$$

$$\text{comp}_{\mathbf{a}}\mathbf{b} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}$$

$$\text{proj}_{\mathbf{a}}\mathbf{b} = \frac{(\mathbf{a}\cdot\mathbf{b})}{(\mathbf{a}\cdot\mathbf{a})}\mathbf{a}$$

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}$$
$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}$$

A line with direction $\mathbf{v}$ passing through the point $\mathbf{r_0}$: $\mathbf{r}(t) = \mathbf{r_0} + t\mathbf{v}$

The equation of the plane tangent to $z = f(x, y)$ at the point $(x_0, y_0, f(x_0, y_0))$:

$$z - f(x_0, y_0) = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

The distance from the point (x_1, y_1, z_1) to the plane $ax + by + cz + d = 0$:

$$\frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

$$s(t) = \int_a^t |\mathbf{r}'(u)|\, du \quad \text{and} \quad \frac{ds}{dt} = |r'(t)|$$

If $F(x, y, z) = K$, then $\dfrac{\partial z}{\partial x} = -\dfrac{F_x}{F_z}$

Curvature for $\mathbf{r}(t)$

$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$$

Curvature for a plane curve $y = f(x)$

$$\kappa = \frac{|f''(x)|}{[1 + (f'(x))^2]^{\frac{3}{2}}}$$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} \qquad \mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} \qquad \mathbf{B}(t) = \mathbf{T}(t)\times\mathbf{N}(t)$$

$D = f_{xx}f_{yy} - f_{xy}^2$

$D > 0$	and $f_{xx} > 0$	implies local minimum
$D > 0$	and $f_{xx} < 0$	implies local maximum
$D < 0$		implies saddle point

CYLINDRICAL: $x = r\cos(\theta)$, $y = r\sin(\theta)$, $z = z$, $dV = r\, dr d\theta dz$

SPHERICAL: $x = \rho\sin(\phi)\cos(\theta)$, $y = \rho\sin(\phi)\sin(\theta)$, $z = \rho\cos(\phi)$, $dV = \rho^2\sin(\phi)\, d\rho d\theta d\phi$

CHANGE OF VARIABLES: $$\iint_R f(x, y)\, dx\, dy = \iint_S f(x(u, v), y(u, v))\left|\frac{\partial(x, y)}{\partial(u, v)}\right| du\, dv$$

$$ds = |r'(t)|dt, \quad dx = x'(t)dt, \quad d\mathbf{r} = \mathbf{r}'(t)dt, \quad dS = |\mathbf{r}_u\times\mathbf{r}_v|\, dA = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA,$$

$$d\mathbf{S} = \mathbf{n}\, dS = \pm(\mathbf{r}_u\times\mathbf{r}_v)\, dA, \quad \mathbf{n} = \pm\frac{\mathbf{r}_u\times\mathbf{r}_v}{|\mathbf{r}_u\times\mathbf{r}_v|} = \pm\frac{\nabla F}{|\nabla F|} = \pm\frac{1}{\sqrt{1 + (f_x)^2 + (f_y)^2}}\langle f_x, f_y, -1\rangle$$

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_S \mathbf{F}\cdot\mathbf{n}\, dS = \iint_D \mathbf{F}\cdot(\mathbf{r}_u\times\mathbf{r}_v)\, dA = \iint_D \left(-P\frac{\partial f}{\partial x} - Q\frac{\partial f}{\partial y} + R\right) dA$$

GREEN'S THEOREM: $$\int_C P\, dx + Q\, dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

STOKES' THEOREM: $$\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \text{curl}\,\mathbf{F}\cdot d\mathbf{S}$$

THE DIVERGENCE THEOREM: $$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_E \text{div}\mathbf{F}\, dV$$