Math 251H Fall 2007 Exam #1 — Answer Key

1. (12 points): Let P, Q, and R be points in $\mathbb{R}^3$.
Using **only** dot products, cross products, and magnitudes, explain how to tell if...

(a) $\vec{PQ}$ and $\vec{PR}$ are orthogonal.

$\vec{PQ}$ and $\vec{PR}$ are orthogonal if and only if $\vec{PQ} \cdot \vec{PR} = 0$.

(b) $\vec{PQ}$ and $\vec{PR}$ are parallel.

$\vec{PQ}$ and $\vec{PR}$ are parallel if and only if $\vec{PQ} \times \vec{PR} = \vec{0}$ (which is equivalent to $|\vec{PQ} \times \vec{PR}| = 0$).

(c) P, Q, and R are colinear (i.e. lie on a common line).

If P, Q, R are on the same line, then $\vec{PQ}$ and $\vec{PR}$ must be parallel. Thus P, Q, and R are colinear if and only if $\vec{PQ} \times \vec{PR} = \vec{0}$.

2. (14 points): Consider the following lines:

$$\text{Line } L_1 : \begin{array}{rcrcr} x(t) & = & 1 & + & t \\ y(t) & = & 1 & + & 2t \\ z(t) & = & 0 & - & t \end{array} \qquad \text{Line } L_2 : \begin{array}{rcrcr} x(t) & = & 3 & - & 2t \\ y(t) & = & 2 & - & 4t \\ z(t) & = & 1 & + & 2t \end{array}$$

(a) Are L_1 and L_2 the same line, parallel lines, intersecting lines, or skew lines?

L_1 is parallel to the vector $\langle 1, 2, -1 \rangle$, and L_2 is parallel to the vector $\langle -2, -4, 2 \rangle$. It's easy to see that $\langle -2, -4, 2 \rangle = -2\langle 1, 2, -1 \rangle$. So the lines are **parallel** (thus not skew).

If there is a point of intersection, then the lines are equal. Let's check for an intersection point:

$$\begin{array}{rcl} 1 + t & = & 3 - 2s \\ 1 + 2t & = & 2 - 4s \\ -t & = & 1 + 2s \end{array}$$

Adding the first and last equations together gives us, $1 = 4$. But it is usually the case that $1 \neq 4$. Thus there is no solution for this system of equations. Hence, the lines do not intersect (so they are not the same).

Answer: These are (distinct) **parallel** lines.

(b) Find a plane which contains the both L_1 and L_2, or explain why this is impossible.

We know that $(1, 1, 0)$ is a point on L_1 (set $t = 0$). Thus our plane contains this point. Now we need a normal vector. To find a normal we will cross 2 vectors which are parallel to the plane. We know that $\langle 1, 2, -1 \rangle$ is parallel to the plane. Also, $\langle -2, -4, 2 \rangle$ is parallel to the plane, but this doesn't help since it's parallel to our first vector. We need to find another vector which is parallel to the plane, but not parallel to the lines. To do this consider $(1, 1, 0)$ on L_1 (set $t = 0$) and $(3, 2, 1)$ on L_2 (set $t = 0$). The vector pointing from $(1, 1, 0)$ to $(3, 2, 1)$ will be parallel to the plane, but not parallel to the lines. This is the vector $\langle 2, 1, 1 \rangle$. Therefore,...

$$\vec{n} = \langle 1, 2, -1 \rangle \times \langle 2, 1, 1 \rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -1 \\ 2 & 1 & 1 \end{vmatrix} = \langle 3, -3, -3 \rangle$$

Answer: $3(x - 1) - 3(y - 1) - 3(z - 0) = 0$

3. (12 points): Find **parametric** equations for the tangent to the curve C at the point $(1, 1, -1)$ where C is parametrized by $\mathbf{r}(t) = \langle t^2 + 1, t + e^t, -\cos(3t)\rangle$

$\mathbf{r}(t) = \langle 1, 1, -1\rangle$ if $t^2+1 = 1$, $t+e^t = 1$, and $-\cos(3t) = -1$. This happens when $t = 0$. To find the direction of this tangent line we need to compute $\mathbf{r}'(0)$. $\mathbf{r}'(t) = \langle 2t, 1 + e^t, 3\sin(3t)\rangle$ so that $\mathbf{r}'(0) = \langle 0, 2, 0\rangle$.

Answer: $$\begin{array}{rcrcr} x(t) & = & & & 1 \\ y(t) & = & 2t & + & 1 \\ z(t) & = & & - & 1 \end{array}.$$

4. (12 points): Suppose that C is a curve which lies on the surface of the unit sphere: $x^2 + y^2 + z^2 = 1$. Let $\mathbf{r}(t)$ (where $a \le t \le b$) be a smooth parametrization of C. Show that $\mathbf{r}(t)\cdot\mathbf{r}'(t) = 0$.

If $\mathbf{r}(t)$ lies on the unit sphere, we must have $|\mathbf{r}(t)| = 1$. Thus $\mathbf{r}(t)\cdot\mathbf{r}(t) = 1^2$. Differentiating both sides, we see that: $\mathbf{r}'(t)\cdot\mathbf{r}(t) + \mathbf{r}(t)\cdot\mathbf{r}'(t) = 0$ (product rule!). Therefore, $2\mathbf{r}(t)\cdot\mathbf{r}'(t) = 0$. Dividing by 2, our result follows.

5. (14 points): Consider $\mathbf{r}(t) = \langle 2\cos(t) + 1, \sqrt{3}\sin(t) + 2, \sin(t) + 3\rangle$.

(a) Compute $\mathbf{T}(t)$, $\mathbf{N}(t)$, $\mathbf{B}(t)$, and the curvature, $\kappa(t)$.

$\mathbf{r}'(t) = \langle -2\sin(t), \sqrt{3}\cos(t), \cos(t)\rangle$, and thus $|\mathbf{r}'(t)| = \sqrt{4\sin^2(t) + 3\cos^2(t) + \cos^2(t)}$ $= \sqrt{4(\sin^2(t) + \cos^2(t))} = \sqrt{4} = 2$.

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \left\langle -\sin(t), \frac{\sqrt{3}}{2}\cos(t), \frac{1}{2}\cos(t)\right\rangle$$

Next, $\mathbf{T}'(t) = \langle -\cos(t), (\sqrt{3}/2)(-\sin(t)), (1/2)(-\sin(t))\rangle$, and thus $|\mathbf{T}'(t)| = \sqrt{\cos^2(t) + (3/4)\sin^2(t) + (1/4)\sin^2(t)} = \sqrt{\cos^2(t) + \sin^2(t)} = 1$.

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \left\langle -\cos(t), -\frac{\sqrt{3}}{2}\sin(t), -\frac{1}{2}\sin(t)\right\rangle$$

Therefore,

$$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin(t) & (\sqrt{3}/2)\cos(t) & (1/2)\cos(t) \\ -\cos(t) & -(\sqrt{3}/2)\sin(t) & -(1/2)\sin(t) \end{vmatrix} = \left\langle 0, -\frac{1}{2}, \frac{\sqrt{3}}{2}\right\rangle$$

Finally,

$$\kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{1}{2}$$

(b) Find the torsion $\tau(t)$. Is this a plane curve? Can you identify this curve?

We could use the complicated formula from the formula sheet to compute $\tau(t)$, but it's easier to use the definition of torsion in this case.

Recall that torsion is defined as follows: $\mathbf{B}'(s) = -\tau(s)\mathbf{N}(s)$ (s is the arc-length parameter). However, $\mathbf{B}(t)$ is a consant vector and so $\mathbf{B}'(t) = \vec{0}$. Therefore, $\tau(t) = 0$. This means that this is a plane curve. Moreover, the curvature is constant (and non-zero), so our curve must be a circle (or part of a circle).

Answer: Our curve is a circle of radius 2 ($= 1/\kappa$) which lies in a plane with normal vector $\mathbf{B} = \langle 0, -1/2, \sqrt{3}/2\rangle$.

6. (12 points): Let $f(x,y) = \begin{cases} \dfrac{x^3y+y^4}{2x^4+3y^4} & (x,y) \neq (0,0) \\ A & (x,y) = (0,0) \end{cases}$ where A is a constant.

(a) Excluding the origin, where is $f(x,y)$ continous? Explain your answer.

$x^3y + y^4$ and $2x^4 + 3y^4$ are polynomials so they are continuous everywhere. Also, if you divide a continuous function by a continuous function, you get a continuous function except at the points where you have divided by 0. But $2x^4 + 3y^4 = 0$ only if both $x = 0$ and $y = 0$ (if either x or y is non-zero, it will contribute a positive number to the sum). Thus the only potential problem spot is the origin.

Answer: $f(x,y)$ is continuous everywhere (except possibly the origin).

(b) Is it possible to choose a value for A which makes $f(x,y)$ continuous at the origin? If so, explain why (and find such an A). If not, explain why not.

For $f(x,y)$ to be continuous at $(0,0)$, we need $\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0) = A$. If we approach the origin along the line $y = x$, we get:

$$\lim_{(x,x)\to(0,0)} f(x,x) = \lim_{x\to 0} \frac{x^3x + x^4}{2x^4 + 3x^4} = \lim_{x\to 0} \frac{2x^4}{5x^4} = \frac{2}{5}$$

On the other hand, if we approach the origin along the line $y = 0$, we get:

$$\lim_{(x,0)\to(0,0)} f(x,0) = \lim_{x\to 0} \frac{x^3(0) + 0^4}{2x^4 + 3(0^4)} = \lim_{x\to 0} 0 = 0$$

So by approaching the origin along two different curves, we get two distinct answers. Therefore, the limit does not exist.

Answer: We cannot make $f(x,y)$ continuous at the origin, since the limit of $f(x,y)$ as (x,y) approaches $(0,0)$ does not exist.

7. (12 points): Let $z = f(x,y)$, $x = r\cos(\theta)$, and $y = r\sin(\theta)$.

(a) Find all of the second partials of $x(r,\theta)$.

The first partials are:

$$\frac{\partial x}{\partial r} = \cos(\theta) \qquad \frac{\partial x}{\partial \theta} = -r\sin(\theta)$$

The second partials are:

$$\frac{\partial^2 x}{\partial r^2} = 0 \qquad \frac{\partial^2 x}{\partial \theta \partial r} = \frac{\partial^2 x}{\partial r \partial \theta} = -\sin(\theta) \qquad \frac{\partial^2 x}{\partial \theta^2} = -r\cos(\theta)$$

(b) Find $\dfrac{\partial z}{\partial \theta}$.

The chain rule gives... $= \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial \theta} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial \theta} = f_x(x,y)(-r\sin(\theta)) + f_y(x,y)(r\cos(\theta))$

Other forms of the answer:

$f_x(r\cos(\theta), r\sin(\theta))(-r\sin(\theta)) + f_y(r\cos(\theta), r\sin(\theta))(r\cos(\theta))$ or $xf_y(x,y) - yf_x(x,y)$

8. (12 points): Let $F(x, y, z) = x^2 - y^2 - z^2$. Consider the level surface $F(x, y, z) = 2$.

(a) Use a directional derivative to show that the vector $\langle 1, 0, 0\rangle$ is not tangent to this level surface at the point $(2, 1, 1)$. Briefly explain your answer.

First, the gradient of $F(x, y, z)$:

$$\nabla F(x, y, z) = \langle F_x, F_y, F_z\rangle = \langle 2x, -2y, -2z\rangle$$

(Note: $\langle 1, 0, 0\rangle$ is a unit vector.)

$$D_{\langle 1,0,0\rangle} F(2, 1, 1) = \nabla F(2, 1, 1)\cdot\langle 1, 0, 0\rangle = \langle 4, -2, -2\rangle\cdot\langle 1, 0, 0\rangle = 4$$

We see that the directional derivative of F at the point $(2, 1, 1)$ in the direction $\langle 1, 0, 0\rangle$ is 4. If $\langle 1, 0, 0\rangle$ were tangent, we should have gotten 0. [Remember $\nabla F(2, 1, 1)$ is normal to the tangent plane, so a dot product with a tangent vector should be 0.]

(b) Find an equation for the plane tangent to this level surface at the point $(2, 1, 1)$.

We have all ready computed the normal vector: $\nabla F(2, 1, 1) = \langle 4, -2, -2\rangle$.

Answer: $4(x - 2) - 2(y - 1) - 2(z - 1) = 0$