Name ________________________________ Math 251, Sec. H1, Exam #1 Oct., 2007

Be sure to show all your work. Unsupported answers will receive no credit.

Use the backs of the exam pages for scratchwork or for continuation of your answers, if necessary.

Problem No.	**Pts Possible**	**Points**
1	12	
2	14	
3	12	
4	12	
5	14	
6	12	
7	12	
8	12	
Total	100	

1. (12 points): Let P, Q, and R be points in $\mathbb{R}^3$.

Using **only** dot products, cross products, and magnitudes, explain how to tell if...

(a) $\vec{PQ}$ and $\vec{PR}$ are orthogonal.

(b) $\vec{PQ}$ and $\vec{PR}$ are parallel.

(c) P, Q, and R are colinear (i.e. lie on a common line).

2. (14 points): Consider the following lines:

$$\text{Line } L_1: \quad \begin{array}{rcrcr} x(t) & = & 1 & + & t \\ y(t) & = & 1 & + & 2t \\ z(t) & = & 0 & - & t \end{array} \qquad\qquad \text{Line } L_2: \quad \begin{array}{rcrcr} x(t) & = & 3 & - & 2t \\ y(t) & = & 2 & - & 4t \\ z(t) & = & 1 & + & 2t \end{array}$$

(a) Are L_1 and L_2 the same line, parallel lines, intersecting lines, or skew lines?

(b) Find a plane which contains the both L_1 and L_2, or explain why this is impossible.

3. (12 points): Find **parametric** equations for the tangent to the curve C at the point $(1, 1, -1)$ where C is parametrized by $\mathbf{r}(t) = \langle t^2 + 1, t + e^t, -\cos(3t)\rangle$

4. (12 points): Suppose that C is a curve which lies on the surface of the unit sphere: $x^2 + y^2 + z^2 = 1$. Let $\mathbf{r}(t)$ (where $a \leq t \leq b$) be a smooth parametrization of C. Show that $\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0$.

5. (14 points): Consider $\mathbf{r}(t) = \langle 2\cos(t) + 1, \sqrt{3}\sin(t) + 2, \sin(t) + 3 \rangle$.

(a) Compute $\mathbf{T}(t)$, $\mathbf{N}(t)$, $\mathbf{B}(t)$, and the curvature, $\kappa(t)$.

(b) Find the torsion $\tau(t)$. Is this a plane curve? Can you identify this curve?

6. (12 points): Let $f(x,y) = \begin{cases} \dfrac{x^3y + y^4}{2x^4 + 3y^4} & (x,y) \neq (0,0) \\ A & (x,y) = (0,0) \end{cases}$ where A is a constant.

(a) Excluding the origin, where is $f(x,y)$ continous? Explain your answer.

(b) Is it possible to choose a value for A which makes $f(x,y)$ continuous at the origin? If so, explain why (and find such an A). If not, explain why not.

7. (12 points): Let $z = f(x, y)$, $x = r\cos(\theta)$, and $y = r\sin(\theta)$.

(a) Find all of the second partials of $x(r, \theta)$.

(b) Find $\dfrac{\partial z}{\partial \theta}$.

8. (12 points): Let $F(x, y, z) = x^2 - y^2 - z^2$. Consider the level surface $F(x, y, z) = 2$.

(a) Use a directional derivative to show that the vector $\langle 1, 0, 0 \rangle$ is not tangent to this level surface at the point $(2, 1, 1)$. Briefly explain your answer.

(b) Find an equation for the plane tangent to this level surface at the point $(2, 1, 1)$.

Math 251H Fall 2007: Formulas for Exam #1

Distance between (x_0, y_0, z_0) and (x_1, y_1, z_1) is: $\sqrt{(x_0 - x_1)^2 + (y_0 - y_1)^2 + (z_0 - z_1)^2}$.

Distance between (x_1, y_1, z_1) and $ax + by + cz + d = 0$ is: $\dfrac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$.

A line with direction $\mathbf{v}$ through a point $\mathbf{r}_0$ is given by: $\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}$.

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos(\theta) \quad \text{and} \quad |\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin(\theta)$$

$$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}, \quad \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}, \quad \text{and} \quad \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$$

The area of a parallelogram spanned by $\mathbf{a}$ and $\mathbf{b}$ is $|\mathbf{a} \times \mathbf{b}|$.
The volume of a parallelepiped spanned by $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ is $|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$.

$$\text{comp}_{\mathbf{a}}(\mathbf{b}) = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|} \quad \text{and} \quad \text{proj}_{\mathbf{a}}(\mathbf{b}) = \frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{a} \cdot \mathbf{a}}\mathbf{a}$$

The linearization of (equation of the tangent plane to) $z = f(x, y)$ at $(x_0, y_0, f(x_0, y_0))$:

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

If $z = f(x_1, x_2, \ldots, x_n)$, then $dz = f_{x_1}dx_1 + f_{x_2}dx_2 + \cdots + f_{x_n}dx_n$.
The tangent plane to $F(x, y, z) = K$ at (a, b, c) is: $\nabla F(a, b, c) \cdot \langle x - a, y - b, z - c \rangle = 0$.

$$\nabla f(x, y, z) = \langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z) \rangle \text{ and } D_{\mathbf{u}} f(x, y, z) = \nabla f(x, y, z) \cdot \mathbf{u}.$$

If $y = f(x)$ is given implicitly by $F(x, y) = K$, then $\dfrac{dy}{dx} = -\dfrac{F_x}{F_y}$.

If $z = f(x, y)$ is given implicitly by $F(x, y, z) = K$, then $\dfrac{\partial z}{\partial x} = -\dfrac{F_x}{F_z}$ and $\dfrac{\partial z}{\partial y} = -\dfrac{F_y}{F_z}$.

Arc length from a to t is given by the function $s(t) = \displaystyle\int_a^t |\mathbf{r}'(u)|\, du$. Thus $s'(t) = |\mathbf{r}'(t)|$.

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} \qquad \mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} \qquad \mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$$

$$\mathbf{T}'(t) = \kappa(t)s'(t)\mathbf{N}(t) \qquad \mathbf{N}'(t) = -\kappa(t)s'(t)\mathbf{T}(t) + \tau(t)s'(t)\mathbf{B}(t) \qquad \mathbf{B}'(t) = -\tau(t)s'(t)\mathbf{N}(t)$$

$$\kappa(t) = \frac{d\mathbf{T}}{ds} = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} \qquad \text{For the curve } y = f(x)\text{: } \kappa(x) = \frac{|f''(x)|}{(1 + (f'(x))^2)^{3/2}}.$$

$$a_T = \frac{\mathbf{r}'(t) \cdot \mathbf{r}''(t)}{|\mathbf{r}'(t)|} \qquad a_N = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|} \qquad \tau(t) = \frac{(\mathbf{r}'(t) \times \mathbf{r}''(t)) \cdot \mathbf{r}'''(t)}{|\mathbf{r}'(t) \times \mathbf{r}''(t)|^2}$$