Math 251H Fall 2007 Exam #2

Answer Key

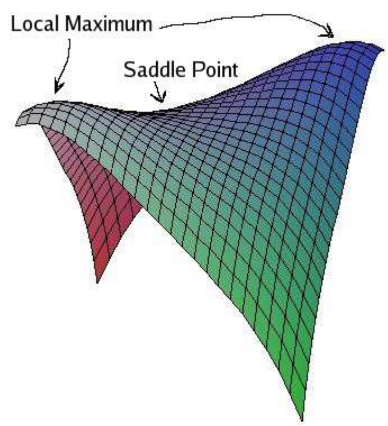
1. (10 points): Find all of the critical points of $f(x,y) = 4xy - x^4 - y^4$. Determine whether each point is a relative minimum, relative maximum, or saddle point.

First, we compute the partial derivates. $f_x(x,y) = 4y - 4x^3$ and $f_y(x,y) = 4x - 4y^3$. So we must solve the system of equations: $4y - 4x^3 = 0$ and $4x - 4y^3 = 0$. Thus $y = x^3$ and $x = y^3$ so that $x = (x^3)^3 = x^9$. The (real) roots of $x^9 - x$ are $x = -1, 0, 1$. Thus $y = (-1)^3 = -1, 0^3 = 0, 1^3 = 1$. So $f(x,y)$ has three critical points $(-1,-1)$, $(0,0)$, and $(1,1)$.

Next, we compute the second partials. $f_{xx}(x,y) = -12x^2$, $f_{yy}(x,y) = -12y^2$, and $f_{xy}(x,y) = f_{yx}(x,y) = 4$.

$(x,y) =$	$f_{xx}(x,y) =$	$f_{yy}(x,y) =$	$f_{xy}(x,y) = f_{yx}(x,y) =$	$D(x,y) =$
$(-1,-1)$	-12	-12	4	$(-12)^2 - 4^2 = 138 > 0$
$(0,0)$	0	0	4	$0^2 - 4^2 = -16 < 0$
$(1,1)$	-12	-12	4	$(-12)^2 - 4^2 = 138 > 0$

Answer:
For $(\pm 1, \pm 1)$ we have $f_{xx} < 0$ and $D > 0$, so these are relative minima.
For $(0,0)$ we have $D < 0$, so this is a saddle point.



2. (10 points): Use the method of "Lagrange multipliers" to find three numbers whose sum is 18 and whose product is as large as possible.

THIS ORIGINAL PROBLEM HAS A FLAW – Consider $A > 0$ and let $B = 18 + 2A$ (> 0). Then $-A - A + B = 18$ and $(-A)(-A)B = A^2(18 + 2A)$ – this product can me made arbitrarily large. Thus there is no maximum! The problem should be replaced with:

"Use the method of "Lagrange multipliers" to find three **non-negative** numbers whose sum is 18 and whose product is as large as possible."

We want the product of x, y, and y to be as large as possible, so we are maximizing $f(x,y,z) = xyz$. We are given the constraint that $g(x,y,z) = x + y + z = 18$. Notice that if x, y, or z is zero, then their product is zero and we *can* make the product positive, so we will ignore these cases and assume that $x > 0$, $y > 0$, and $z > 0$.

First note that $\nabla f(x,y,z) = \langle yz, xz, xy\rangle$ and that $\nabla g(x,y,z) = \langle 1,1,1\rangle$. Using a Lagrange multiplier, λ, we get the vector equation: $\langle yz, xz, xy\rangle = \lambda\langle 1,1,1\rangle$. So we must solve the following system of equations:

$$\begin{aligned} yz &= \lambda \\ xz &= \lambda \\ xy &= \lambda \\ x+y+z &= 18 \end{aligned}$$

Note: Keep in mind, we are assuming that x, y, and z are positive.

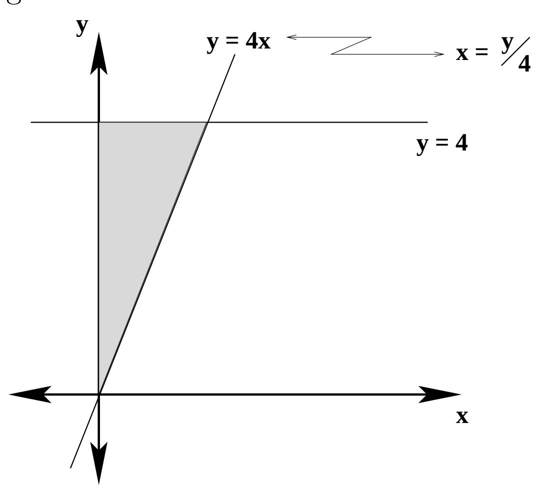
Setting the first two equations together, we get that $yz = xz$. Since z is not zero, we cancel it off and get that $y = x$. Likewise, $xz = xy$, x is not zero, cancelling we get $z = y$. Therefore, $x = y = z$ and thus $3x = x + y + z = 18$ so that $x = y = z = 6$.

Our only critical point (with no zero coordinates) is $(6,6,6)$. Since the critical points with zeros give $f(x,y,z) = 0$, we conclude that $f(6,6,6) = 6^3 = 216$ is the maximum.

Answer: $6,6,6$ sum up to 18 and give the maximal product of $6^3 = 216$.

3. (12 points): Consider the integral: $\displaystyle\int_0^1 \int_{4x}^4 e^{-y^2}\,dy\,dx$.

(a) Sketch the region of integration.



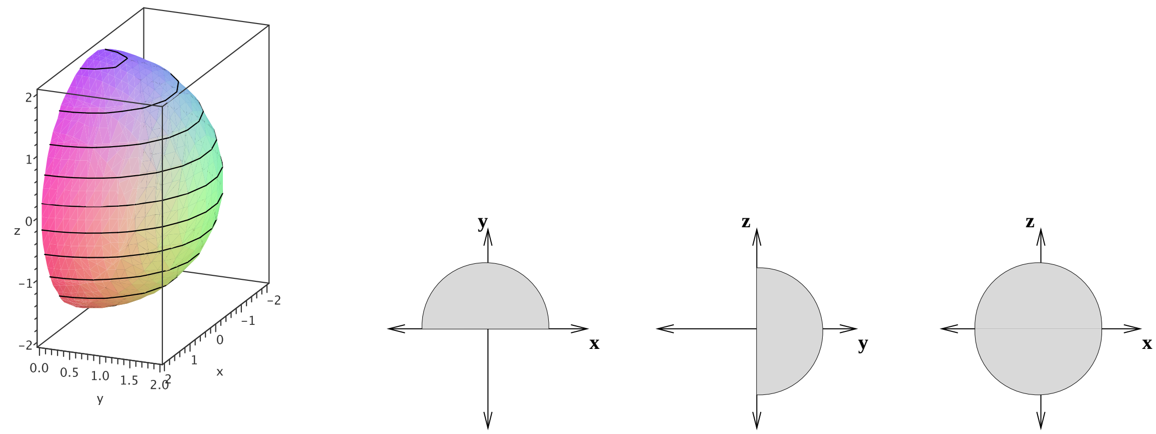
(b) Evaluate the integral.

We cannot integrate e^{-y^2} with respect to y, so we will reverse the order of integration. Notice (as a type II region) the left boundary of the region is $x = 0$ (the y-axis) and the right boundary is $x = y/4$. Then y ranges from 0 up to 4. Thus we have that:

$$\begin{aligned} \int_0^1 \int_{4x}^4 e^{-y^2}\,dy\,dx &= \int_0^4 \int_0^{y/4} e^{-y^2}\,dx\,dy = \int_0^4 xe^{-y^2}\Big|_0^{y/4}\,dy = \int_0^4 \frac{1}{4}ye^{-y^2}\,dy \\ &= -\frac{1}{8}e^{-y^2}\Big|_0^4 = -\frac{1}{8}e^{-4^2} - \left(-\frac{1}{8}e^{-0^2}\right) = \frac{1}{8}\left(1 - e^{-16}\right) \end{aligned}$$

4. (13 points): Consider the triple integral $\iiint_E f(x,y,z)\,dV$ where E is the solid inside the sphere $x^2+y^2+z^2=4$ and to the right of $y=0$ (that is $y \geq 0$).

The region of integration and corresponding projections onto the xy, yz, and xz-planes:



(a) Express the above triple integral as an interated integral in the following orders of integration:

The top and bottom of the region are given by $z = \pm\sqrt{4-x^2-y^2}$. Projecting onto the xy-plane we get a half disk $x^2+y^2 \leq 4$, $y \geq 0$. So its top is $y=\sqrt{4-x^2}$ and bottom is $y=0$. x should range from -2 to 2. Our integral is:

$$\int_{-2}^{2}\int_{0}^{\sqrt{4-x^2}}\int_{-\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}} f(x,y,z)\,dz\,dy\,dx$$

The front and back of the region are given by $x = \pm\sqrt{4-y^2-z^2}$. Projecting onto the yz-plane we get a half disk $y^2+z^2 \leq 4$, $y \geq 0$. So its left side is $y=0$ and right side is $y=\sqrt{4-z^2}$. z should range from -2 to 2. Our integral is:

$$\int_{-2}^{2}\int_{0}^{\sqrt{4-z^2}}\int_{-\sqrt{4-y^2-z^2}}^{\sqrt{4-y^2-z^2}} f(x,y,z)\,dx\,dy\,dz$$

The left side of the region is given by $y=0$ and the right side is given by $y=\sqrt{4-x^2-z^2}$. Projecting onto the xz-plane we get a disk $x^2+z^2 \leq 4$. So its top and bottom are $z=\pm\sqrt{4-x^2}$. x should range from -2 to 2. Our integral is:

$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{0}^{\sqrt{4-x^2-z^2}} f(x,y,z)\,dy\,dz\,dx$$

(b) Rewrite the integral in cylindrical coordinates – *this answer should not have any x's or y's!*

Looking at the first integral in part (a), we see that the z bounds should be $\pm\sqrt{4-r^2}$. After integrating out z, we are left with a half-disk in the xy-plane. This corresponds to $0 \leq r \leq 2$ and $0 \leq \theta \leq \pi$ (just go half way around). Our integral is:

$$\int_{0}^{\pi}\int_{0}^{2}\int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} f(r\cos(\theta), r\sin(\theta), z)r\,dz\,dr\,d\theta$$

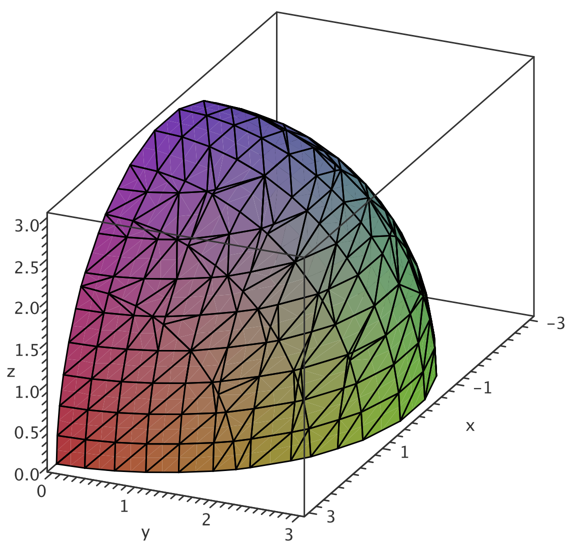
(c) Rewrite the integral in spherical coordinates – *this answer should not have any x's, y's, or z's!*

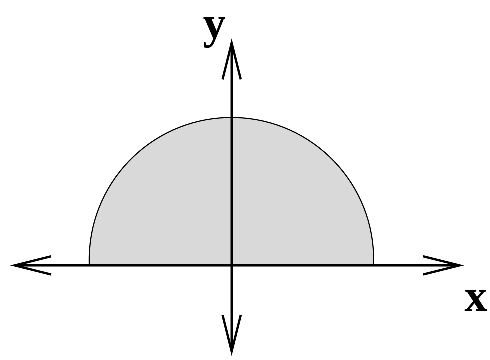
This half of the sphere (of radius 2) is given by $0 \le \rho \le 2$, $0 \le \theta \le \pi$ (as with cylindrical coordinates), and $0 \le \phi \le \pi$ (we want the top and bottom halves). Our integral is:

$$\int_0^{\pi}\int_0^{\pi}\int_0^{2} f(\rho\cos(\theta)\sin(\phi), \rho\sin(\theta)\sin(\phi), \rho\cos(\phi))\rho^2\sin(\phi)\,d\rho\,d\phi\,d\theta$$

5. (12 points): Evaluate $\displaystyle\int_{-3}^{3}\int_0^{\sqrt{9-x^2}}\int_0^{\sqrt{9-x^2-y^2}} \frac{1}{\sqrt{x^2+y^2+z^2}}\,dz\,dy\,dx$

The region of integration along with its projection onto the xy-plane:



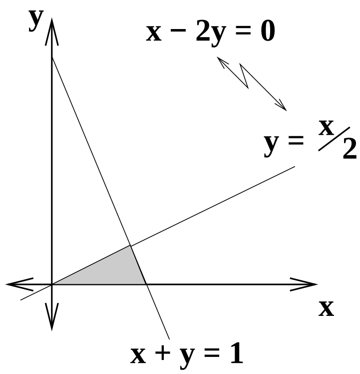


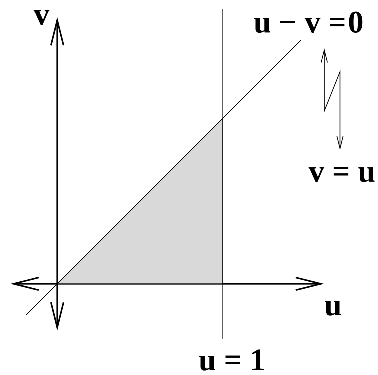
Switch to spherical coordinates. Notice we have $0 \le z \le \sqrt{9-x^2-y^2}$ (the top half of a sphere) and $0 \le y \le \sqrt{9-x^2}$ (the top half of a circle) and $-3 \le x \le 3$ (both sides of the circle). Since we have a sphere of radius 3, we let $0 \le \rho \le 3$. We want just the top half, so $0 \le \phi \le \frac{\pi}{2}$. When projected into the xy-plane, we have just the top half of the circle, so $0 \le \theta \le \pi$ (just half way around). Therefore, we get:

$$= \int_0^{\pi}\int_0^{\pi/2}\int_0^{3} \frac{1}{\rho}\rho^2\sin(\phi)\,d\rho\,d\phi\,d\theta = \int_0^{\pi} d\theta \int_0^{\pi/2}\sin(\phi)\,d\phi \int_0^{3}\rho\,d\rho = \pi\cdot 1\cdot\frac{3^2}{2} = \frac{9\pi}{2}$$

6. (14 points): Let D be the region bounded by the lines: $y = x/2$, $x + y = 1$, and $y = 0$. Evaluate the double integral $\displaystyle\iint_D \sqrt{\frac{x+y}{x-2y}}\,dA$ by changing variables. Please include sketches of both the region D and the new region obtained after changing variables. *Don't forget the Jacobian!*

The natural choice for changing variables is: $u = x + y$ and $v = x - 2y$ since this will not only simplify what we are integrating, but also transform our region of integration into something nice. Notice that the bounds of integration are the lines: $x - 2y = 0$, $x + y = 1$, and $y = 0$. In terms of u and v these become: $v = 0$, $u = 1$, and $u - v = 0$ (since $u - v = (x + y) - (x - 2y) = 3y = 3(0) = 0$). Here is a sketch of these two regions:





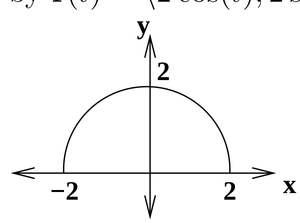
Solving $u = x + y$ and $v = x - 2y$ for x and y, we get $x = \frac{2}{3}u + \frac{1}{3}v$ and $y = \frac{1}{3}u - \frac{1}{3}v$. So that $\dfrac{\partial(x, y)}{\partial(u, v)} = \begin{bmatrix} 2/3 & 1/3 \\ 1/3 & -1/3 \end{bmatrix} \xrightarrow{\text{det}} \dfrac{2}{3}\left(-\dfrac{1}{3}\right) - \left(\dfrac{1}{3}\right)^2 = -2/9 - 1/9 = -1/3$. So we get the following:

$$\begin{aligned} \iint_D \sqrt{\frac{x+y}{x-2y}}\, dA &= \int_0^1 \int_0^u \sqrt{\frac{u}{v}} \left| -\frac{1}{3} \right| dv\, du = \int_0^1 \int_0^u \frac{1}{3} u^{1/2} v^{-1/2}\, dv\, du \\ &= \int_0^1 \frac{2}{3} u^{1/2} v^{1/2} \Big|_0^u du = \int_0^1 \frac{2}{3} u\, du = \frac{u^2}{3}\Big|_0^1 = \frac{1}{3} \end{aligned}$$

7. (14 points): A wire is bent into a semicircular shape described by $x^2 + y^2 = 4$, $-2 \le x \le 2$, and $y \ge 0$. Suppose that the wire has constant density (set $\rho = 1$).

(a) Parameterize and sketch the curve $x^2 + y^2 = 4$, $-2 \le x \le 2$. Then determine it's length two ways: (i) Using highschool geometry and (ii) Using a line integral.

We can parametrize our curve by $\mathbf{r}(t) = \langle 2\cos(t), 2\sin(t) \rangle$ where $0 \le t \le \pi$.



(i) The circumference of a circle is given by $2\pi r$ where r is the radius of a circle. We have half of a circle of radius 2, so the answer is $\frac{2\pi(2)}{2} = 2\pi$.

(ii) We need to compute $\int_C 1\, ds$ (this gives the arc length of C) where C is our semicircle. $\mathbf{r}'(t) = \langle -2\sin(t), 2\cos(t) \rangle$ and thus $|\mathbf{r}'(t)| = 2$.

$$\int_C 1\, ds = \int_0^\pi |\mathbf{r}'(t)|\, dt = \int_0^\pi 2\, dt = 2\pi$$

(b) Find the wire's center of mass.

$$M_y = \int_C x\, ds = \int_0^\pi 2\cos(t) 2\, dt = 4\sin(t)|_0^\pi = 0$$

$$M_x = \int_C y\, ds = \int_0^\pi 2\sin(t) 2\, dt = -4\cos(t)|_0^\pi = (-4)(-1) - (-4)(1) = 8$$

$$\textbf{Answer}: \quad (\bar{x}, \bar{y}) = \frac{1}{m}(M_y, M_x) = \frac{1}{2\pi}(0, 8) = \left(0, \frac{4}{\pi}\right)$$

8. (15 points): Let $\mathbf{F}(x, y, z) = \langle 2x + yz\cos(x), z\sin(x) + z^2, y\sin(x) + 2yz + 3z^2\rangle$.

(a) Is $\mathbf{F}(x, y, z)$ conservative? If so, find a potential function for $\mathbf{F}(x, y, z)$. If not, explain why $\mathbf{F}(x, y, z)$ is not conservative.

Let $\mathbf{F} = \langle P, Q, R\rangle$. To show $\mathbf{F}$ is not conservative, we should look for $P_y \neq Q_x$, $P_z \neq R_x$, or $Q_z \neq R_y$ but for our $\mathbf{F}$ all of these are equal, so we suspect $\mathbf{F}$ is conservative.

$$\begin{aligned}
\int 2x + yz\cos(x)\,dx &= x^2 + yz\sin(x) + g_1(y, z) \\
\int z\sin(x) + z^2\,dy &= yz\sin(x) + yz^2 + g_2(x, z) \\
\int y\sin(x) + 2yz + 3z^2\,dz &= yz\sin(x) + y^2z + z^3 + g_3(x, y)
\end{aligned}$$

Answer: $f(x, y, z) = x^2 + yz\sin(x) + yz^2 + z^3$ (plus a constant, if you want). Since $\nabla f = \mathbf{F}$, $\mathbf{F}$ is conservative.

(b) Let C be the curve $\mathbf{r}(t) = \langle t, \sin(t), \cos(t) - 1\rangle$ where $0 \leq t \leq 2\pi$. Compute $\int_C \mathbf{F}\cdot d\mathbf{r}$ where $\mathbf{F}$ is the vector field from part (a).

Notice that $\mathbf{r}(0) = \langle 0, 0, 0\rangle$ and $\mathbf{r}(2\pi) = \langle 2\pi, 0, 0\rangle$. Since $\mathbf{F}$ is conservative, we can use the fundamental theorem of line integrals:

$$\int_C \mathbf{F}\cdot d\mathbf{r} = f(\mathbf{r}(2\pi)) - f(\mathbf{r}(0)) = f(2\pi, 0, 0) - f(0, 0, 0) = (2\pi)^2 - 0 = 4\pi^2$$

(c) Let $\mathbf{G}(x, y) = \langle P(x, y), Q(x, y)\rangle$ be a conservative vector field and let C be some **closed curve** such that $\int_C P(x, y)\,dx = 5$. Find $\int_C Q(x, y)\,dy$ and explain your answer.

Since $\mathbf{G}$ is conservative, $\int_C \mathbf{G}\cdot d\mathbf{r} = 0$ for any closed curve (by the fund. thm. of line integrals). Therefore,

$$0 = \int_C \mathbf{G}\cdot d\mathbf{r} = \int_C P\,dx + Q\,dy = \int_C P\,dx + \int_C Q\,dy = 5 + \int_C Q\,dy$$

Answer: $\int_C Q\,dy = -5$.