Math 291 Spring 2006
Exam #1 Answer Key

1. (12 points): "Thank you Dr. Cook! This problem was a gift."

(a) Consider the points $(1, 2, 3)$, $(4, 5, 6)$, and $(1, 1, 1)$. Find the equation of the plane which contains these points.

The vector from $(1, 2, 3)$ to $(4, 5, 6)$: $\langle 4-1, 5-2, 6-3\rangle = \langle 3, 3, 3\rangle$ is parallel to this plane. Likewise, the vector from $(1, 2, 3)$ to $(1, 1, 1)$: $\langle 1-1, 1-2, 1-3\rangle = \langle 0, -1, -2\rangle$ is also parallel. Thus we get a normal vector by taking their cross product:

$$\begin{aligned}
\langle 3,3,3\rangle \times \langle 0,-1,-2\rangle &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 3 & 3 \\ 0 & -1 & -2 \end{vmatrix} \\
&= \langle 3(-2)-(-1)3, -(3(-2)-0(3)), 3(-1)-0(3)\rangle \\
&= \langle -3, 6, -3\rangle.
\end{aligned}$$

We know that the plane goes through the point $(1, 2, 3)$, so we get

Answer: $-3(x-1)+6(y-2)-3(z-3)=0$ that is $-3x+6y-3z=0$.

(b) Find parametric equations for a line which is perpendicular to the plane found in part (a) and which intersects that plane at the point $(1, 2, 3)$.

Since the line is perpendicular to the plane, its direction vector should be parallel to a normal vector for the plane: $\langle -3, 6, -3\rangle$. Also, we know that the line passes through $(1, 2, 3)$. Therefore,

Answer: $x(t) = -3t+1$, $y(t) = 6t+2$, $z(t) = -3t+3$ (Of course there are many different ways to write down the answer to this problem.)

(c) Find the area of the parallelogram with vertices: $(1, 2, 3)$, $(4, 5, 6)$, $(1, 1, 1)$, and $(4, 4, 4)$.

The vector from $(1, 2, 3)$ to $(4, 5, 6)$ is $\langle 3, 3, 3\rangle$. The vector from $(1, 2, 3)$ to $(1, 1, 1)$ is $\langle 0, -1, -2\rangle$. Notice that starting at $(1, 2, 3)$ and travelling along $\langle 3, 3, 3\rangle$ and then along $\langle 0, -1, -2\rangle$ we arrive at $(1+3+0, 2+3-1, 3+3-2) = (4, 4, 4)$ (the other corner).
Therefore, this parallelogram is spanned by $\langle 3, 3, 3\rangle$ and $\langle 0, -1, -2\rangle$. Thus its area is equal to the length of the cross product of these two vectors. We all ready computed in part this cross product in part (a).

Answer: The area is $|\langle -3, 6, -3\rangle| = \sqrt{(-3)^2+6^2+(-3)^2} = \sqrt{54} = 3\sqrt{6}$.

2. (13 points): Choose **one** of the following:

I. Given a vector $\mathbf{a} \neq \mathbf{0}$ such that $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c}$ and $\mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{c}$, show that we must have $\mathbf{b} = \mathbf{c}$.

Notice that $\mathbf{a} \cdot (\mathbf{b} - \mathbf{c}) = 0$. Thus $\mathbf{a}$ is perpendicular to $(\mathbf{b} - \mathbf{c})$. Also, $\mathbf{a} \times (\mathbf{b} - \mathbf{c}) = \mathbf{0}$. Thus $\mathbf{a}$ is parallel to $(\mathbf{b} - \mathbf{c})$. Since $\mathbf{a}$ and $(\mathbf{b} - \mathbf{c})$ are simultaneously perpendicular and parallel, we must have that either $\mathbf{a} = \mathbf{0}$ or $\mathbf{b} - \mathbf{c} = \mathbf{0}$. But we assumed that $\mathbf{a} \neq \mathbf{0}$. Thus $\mathbf{b} - \mathbf{c} = \mathbf{0}$ and so $\mathbf{b} = \mathbf{c}$.

II. Given that $\mathbf{r}(t) \neq 0$ and $\mathbf{r}(t)$ is differentiable, show that

$$\frac{d}{dt}|\mathbf{r}(t)| = \frac{1}{|\mathbf{r}(t)|}\mathbf{r}(t) \cdot \mathbf{r}'(t).$$

$$\frac{d}{dt}|\mathbf{r}(t)| = \frac{d}{dt}\sqrt{|\mathbf{r}(t)|^2} = \frac{d}{dt}\sqrt{\mathbf{r}(t) \cdot \mathbf{r}(t)}$$

By the dot product product rule and the chain rule, we get:

$$\begin{aligned} \frac{d}{dt}\left(\mathbf{r}(t) \cdot \mathbf{r}(t)\right)^{1/2} &= \frac{1}{2}\left(\mathbf{r}(t) \cdot \mathbf{r}(t)\right)^{-1/2}\left(\mathbf{r}'(t) \cdot \mathbf{r}(t) + \mathbf{r}(t) \cdot \mathbf{r}'(t)\right) \\ &= \frac{1}{2\sqrt{|\mathbf{r}(t)|^2}} 2\mathbf{r}(t) \cdot \mathbf{r}'(t) \\ &= \frac{1}{|\mathbf{r}(t)|}\mathbf{r}(t) \cdot \mathbf{r}'(t). \end{aligned}$$

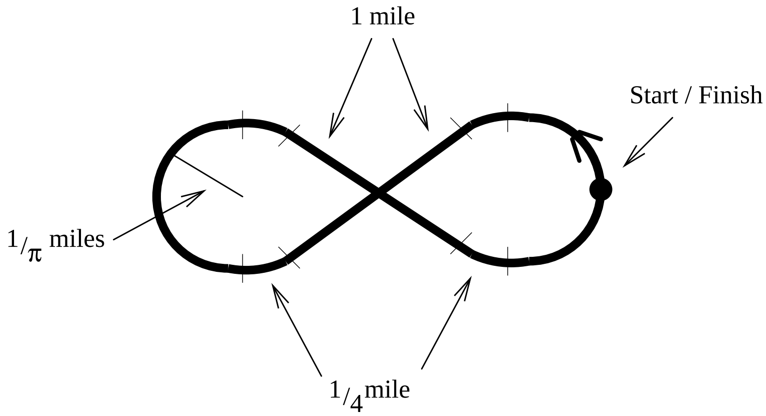
3. (14 points): Some differential geometry.

(a) I am a curve with a smooth parametrization $\mathbf{r}(t)$. If you compute my $\mathbf{B}(t)$ vector, you will find that it is a (well-defined, non-zero) constant vector for all t. My curvature, $\kappa(t)$, is constant for all t. What am I?

If $\mathbf{B}(t)$ is constant, then the curve must lie in a plane (with normal vector $\mathbf{B}(t) = \mathbf{B}$). If a curve in a plane has constant curvature, it must be part of a line or part of a circle. However, if the mystery curve was a line, then $\mathbf{T}(t)$ would be constant (lines have a constant tangent vector). But then $\mathbf{T}'(t) = 0$ and thus $\mathbf{N}(t)$ and $\mathbf{B}(t)$ are not well-defined. Therefore,

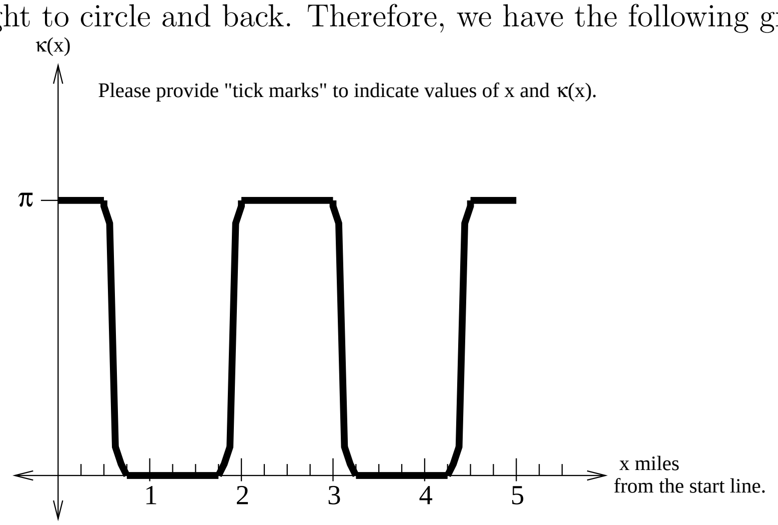
Answer: The mystery curve is a part (or all) of a circle whose radius is $1/\kappa$ which lies in a plane with normal vector $\mathbf{B}$.

(b) Here is a diagram of "Bill's Less Than Safe Speedway" (it's not exactly safe because it crosses itself in the middle).



Use this diagram, to sketch the speedway's curvature on the axes below.

Notice that the speed way has 2 1-mile straightways, 4 1/4-mile segments, and 2 half circles of radius $1/\pi$. Thus the track is $2(1) + 4(1/4) + 2(1/\pi)\pi = 5$ miles long. Curvature on the straightways is 0. Curvature on a semicircle of radius $1/\pi$ is π (a circle of radius a has curvature $1/a$). The 1/4-mile segments transition from straight to circle and back. Therefore, we have the following graph:



(c) Give your best guess as to what $\tau(t)$ should be for my speedway. And (as always) explain your choice.

Since the raceway seems to be flat, the track is a planar curve and thus torsion is 0.

Answer: $\tau(t) = 0$

4. (14 points): Continuity $\neq$ Fun. Proof: Given any $\epsilon > 0$ there exists pain > 0...

(a) Can you find a number A which makes

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & (x,y) \neq (0,0) \\ A & (x,y) = (0,0) \end{cases}$$

continuous everywhere?

If not, explain why. If so, find A and show that f is continous everywhere.

Let's examine the limit: $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$. The limit toward the origin along the line $y = 0$ is $\displaystyle\lim_{x\to 0} \frac{x(0)}{x^2+0^2} = \lim_{x\to 0} 0 = 0$. However, if we approach the origin along the line $y = x$ we get $\displaystyle\lim_{x\to 0} \frac{x(x)}{x^2+x^2} = \lim_{x\to 0} \frac{x^2}{2x^2} = \lim_{x\to 0} \frac{1}{2} = \frac{1}{2}$. Since we get different answer along different curves, this limit does not exit. Therefore, there is no hope of $f(x,y)$ being continuous at the origin.

Answer: There is no way to make $f(x,y)$ continous everywhere, since the limit of $f(x,y)$ as (x,y) approaches $(0,0)$ does not exist.

(b) Can you find a number A which makes

$$f(x,y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}} & (x,y) \neq (0,0) \\ A & (x,y) = (0,0) \end{cases}$$

continuous everywhere?

If not, explain why. If so, find A and show that f is continous everywhere.

Now $x^2 + y^2 > 0$ and thus $\sqrt{x^2+y^2} > 0$ when $(x,y) \neq (0,0)$. Thus since $f(x,y)$ is built up from continuous functions away from the origin, we have the $f(x,y)$ is continuous everywhere except possibly at $(0,0)$.

Now let's examine the limit: $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}}$. Switching to polar coordinates, we get ($x = r\cos(\theta)$, $y = r\sin(\theta)$, $\sqrt{x^2+y^2} = r$):

$$\lim_{(r,\theta)\to(0,c)} \frac{r\cos(\theta) r\sin(\theta)}{r} \lim_{(r,\theta)\to(0,c)} r\cos(\theta)\sin(\theta) = 0\cos(c)\sin(c) = 0.$$

Thus if we let $A = 0$, then we have $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0)$. Therefore, $f(x,y)$ with $A = 0$ is continous at the origin.

Answer: $f(x,y)$ is continuous everywhere if (and only if) we let $A = 0$.

5. (14 points): Choose **one** of the following:

I. Given $z = f(x, y)$ is a differentiable function, $x(s, t) = s^2+t^2$, and $y(s, t) = s^2-t^2$ show that

$$\frac{\partial z}{\partial s}\frac{\partial z}{\partial t} = 4st\left(\left(\frac{\partial z}{\partial x}\right)^2 - \left(\frac{\partial z}{\partial y}\right)^2\right)$$

Since we are dealing with differentiable functions, the chain rule applies. We have

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = \frac{\partial z}{\partial x}2s + \frac{\partial z}{\partial y}2s = 2s\left(\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\right)$$

and

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = \frac{\partial z}{\partial x}2t + \frac{\partial z}{\partial y}(-2t) = 2t\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)$$

Multiplying these together we get:

$$\frac{\partial z}{\partial s}\frac{\partial z}{\partial t} = 2s\left(\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\right)2t\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right) = 4st\left(\left(\frac{\partial z}{\partial x}\right)^2 + \frac{\partial z}{\partial x}\frac{\partial z}{\partial y} - \frac{\partial z}{\partial x}\frac{\partial z}{\partial y} + \left(\frac{\partial z}{\partial x}\right)^2\right)$$

Therefore,

$$\frac{\partial z}{\partial s}\frac{\partial z}{\partial t} = 4st\left(\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial x}\right)^2\right)$$

II. State and prove the chain rule for $z = f(x, y)$, $x = g(t)$, and $y = h(t)$ where f, g, and h are differentiable everywhere.

The chain rule (in this case) states that

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial x}\frac{dx}{dt}.$$

We know that f, g, and h are differentiable. Thus there exists ϵ_1, ϵ_2, ϵ^g, and ϵ_h such that

- $f(x+a, y+b) = f(x, y) + f_x(x, y)a + f_x(x, y)b + \epsilon_1 a + \epsilon_2 b = f(x, y) + (\epsilon_1 + f_x(x, y))a + (\epsilon_2 + f_x(x, y))b$ where $\epsilon_1, \epsilon_2 \to 0$ as $(a, b) \to (0, 0)$.
- $g(t+l) = g(t) + g'(t)l + \epsilon^g l = g(t) + (\epsilon^g + g'(t))l$ where $\epsilon^g \to 0$ as $l \to 0$.
- $h(t+l) = h(t) + h'(t)l + \epsilon^h l = h(t) + (\epsilon^h + h'(t))l$ where $\epsilon^h \to 0$ as $l \to 0$.

For convenience, let $f(g(t), h(t)) = A$

$$\begin{aligned}
\frac{dz}{dt} &= \lim_{l\to 0} \frac{f(g(t+l), h(t+l)) - f(g(t), h(t))}{l} \\
&= \lim_{l\to 0} \frac{f(g(t) + (\epsilon^g + g'(t))l, h(t) + (\epsilon^h + h'(t))l) - A}{l} \\
&= \lim_{l\to 0} \frac{A + (\epsilon_1 + f_x(g(t), h(t)))(\epsilon^g + g'(t))l + (\epsilon_2 + f_y(g(t), h(t)))(\epsilon^h + h'(t))l - A}{l} \\
&= \lim_{l\to 0} \frac{l((\epsilon_1 + f_x(g(t), h(t)))(\epsilon^g + g'(t)) + (\epsilon_2 + f_y(g(t), h(t)))(\epsilon^h + h'(t)))}{l} \\
&= \lim_{l\to 0} (\epsilon_1 + f_x(g(t), h(t)))(\epsilon^g + g'(t)) + (\epsilon_2 + f_y(g(t), h(t)))(\epsilon^h + h'(t))
\end{aligned}$$

Notice that as $l \to 0$, we have not only $\epsilon^g, \epsilon^h \to 0$ but also $(\epsilon^g + g'(t))l, (\epsilon^h + h'(t))l) \to (0, 0)$ and thus $\epsilon_1, \epsilon_2 \to 0$. Therefore,

$$\begin{aligned}
\frac{dz}{dt} &= \lim_{l\to 0} (\epsilon_1 + f_x(g(t), h(t)))(\epsilon^g + g'(t)) + (\epsilon_2 + f_y(g(t), h(t)))(\epsilon^h + h'(t)) \\
&= (0 + f_x(g(t), h(t)))(0 + g'(t)) + (0 + f_y(g(t), h(t)))(0 + h'(t)) \\
&= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial x}\frac{dx}{dt}
\end{aligned}$$

6. (20 points): An odd collection of problems. Let $F(x, y, z) = xy - z^2$.

(a) Find the equation of the plane tangent to $F(x, y, z) = 1$ at $(2, 1, -1)$.

We know that the gradient gives us normals for tangent planes to level surfaces. $\nabla F(x, y, z) = \langle y, x, -2z\rangle$. So a normal for the tagent plane at $(2, 1, -1)$ is $\nabla F(2, 1, -1) = \langle 1, 2, 2\rangle$. This plane is tangent at the point $(2, 1, -1)$ (and thus contains this point).

Answer: $1(x - 2) + 2(y - 1) + 2(z - (-1)) = 0$ that is $x + 2y - 2z - 2 = 0$.

(b) Compute $\frac{\partial z}{\partial x}$ given $xy - z^2 = 1$.

Implicit differentiation: $\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}$. We have all ready found that $F_x(x, y, z) = y$ and $F_z(x, y, z) = -2z$.

Answer: $\frac{\partial z}{\partial x} = \frac{y}{2z}$

(c) What point(s) of the surface $F(x, y, z) = 1$ are closest to the origin?

We want to minimize the distance to the origin (i.e. $D = \sqrt{x^2 + y^2 + z^2}$) given the constraint of staying on the surface $F(x, y, z) = 1$. Equivalently we can minimize the square of the distance to the origin (this will give us the same answer while making computations simpler). We will use the method of Lagrange multipliers. Note: we are minimizing $f(x, y, z) = x^2 + y^2 + z^2$ $(= D^2)$ subject to the constraint $F(x, y, z) = xy - z^2 = 1$. We all ready know that $\nabla F(x, y, z) = \langle y, x, -2z \rangle$. It is easy to find that $\nabla f(x, y, z) = \langle 2x, 2y, 2z \rangle$. Therefore we have $\langle y, x, -2z \rangle = \lambda \langle 2x, 2y, 2z \rangle$. This along with the constraint gives us the following system of equations:

$$\begin{aligned} y &= \lambda 2x \\ x &= \lambda 2y \\ -2z &= \lambda 2z \\ 1 &= xy - z^2 \end{aligned}$$

Notice that if $z \neq 0$ we have that $-2 = \lambda 2$ thus $\lambda = -1$. This means that $y = -2x$ and $x = -2y$. Therefore, $y = -2(-2y) = 4y$. Thus $x = 0$ and $y = 0$. Using the constraint equation we see that $1 = 0 - z^2$ which is $z^2 = -1$. So there are no real solutions, if $z \neq 0$.
On the other hand, if $z = 0$, we have $xy = 1$. thus x and y must be non-zero. Thus $2\lambda = y/x$ and $2\lambda = x/y$. Hence, $x^2 = y^2$ so that $x = \pm y$. But $xy = 1$ implies that x and y have the same sign. thus $xy = x^2 = 1$. Therefore, $x = y = \pm 1$.
Finally, substituting these values into D we get: $D(1, 1, 0) = D(-1, -1, 0) = \sqrt{2}$.
Note: There is no maximum distance from the origin. Since if $z = 0$ and $y = 1/x$, then $xy - z^2 = 1$ and $D(x, y, z) = D(x, 1/x, 0) = \sqrt{x^2 + (1/x)^2}$ which we can make arbitrarily large.

Answer: $(1, 1, 0)$ and $(-1, -1, 0)$ are the two points on $xy - z^2 = 1$ which are closest to the origin.

7. (13 points): Loose ends.

(a) Find and classify all critical points of $z = -x^3 + 4xy - 2y^2 + 1$.

$\dfrac{\partial z}{\partial x} = -3x^2 + 4y$ and $\dfrac{\partial z}{\partial y} = 4x - 4y$

Thus if $\dfrac{\partial z}{\partial y} = 4x - 4y = 0$, we must have $x = y$. Then if in addtion $\dfrac{\partial z}{\partial x} = -3x^2 + 4y = 0$, we must have $-3x^2 + 4x = 0$. Thus $3x^2 = 4x$. So either $x = 0$ or $x = 4/3$.
Our two cricital points are $(0,0)$ and $(4/3, 4/3)$.
To classify these points we need to find the second partial derivatives also. $\dfrac{\partial^2 z}{\partial x^2} = -6x$, $\dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{\partial^2 z}{\partial y \partial x} = 4$, and $\dfrac{\partial^2 z}{\partial y^2} = -4$
Thus $D(x,y) = -6x(-4) - 4^2 = 24x - 16$. $D(0,0) = -16 < 0$ (saddle point). $D(4/3,4/3) = 24(4/3) - 16 = 32 - 16 = 16 > 0$ and $f_{xx}(4/3,4/3) = -6(4/3) = -8 < 0$ (relative maximum).

Answer: $z = -x^3 + 4xy - 2y^2 + 1$ has a saddle point at $(0,0)$ and a relative maximum at $(4/3, 4/3)$.

(b) I am standing on the surface $z = -x^3 + 4xy - 2y^2 + 1$ at the point $(1,2,0)$. I want to climb up hill in the fastest possible manner. What direction should I face before I start climbing (give your answer in the form of a **unit** vector $\mathbf{u} = \langle a, b \rangle$)?

To climb up hill in the fastest possible manner, I need to face in the direction of the gradient vector (remember the gradient vector direction maximizes the directional derivative). Let $f(x,y) = -x^3 + 4xy - 2y^2 + 1$. Then $\nabla f(x,y) = \langle -3x^2 + 4y, 4x - 4y \rangle$. Thus the gradient vector at the point $(1,2,0)$ is $\nabla f(1,2) = \langle -3(1^2) + 4(2), 4(1) - 4(2) \rangle = \langle 5, -4 \rangle$. But we want a unit vector so we must divide by its length $|\nabla f(1,2)| = \sqrt{5^2 + (-4)^2} = \sqrt{25 + 16} = \sqrt{41}$.

Answer: $\mathbf{u} = \dfrac{1}{\sqrt{41}} \langle 5, -4 \rangle$

(c) In order to move 1 unit horizontally in the direction found in part (b), approximately how far up will I have to climb?

If I change my (x,y)-coordinates from (a,b) to $(a,b) + h\mathbf{u}$ then my z-coordinate should change from c to approximately $c + hD_{\mathbf{u}} f(a,b)$ ($D_{\mathbf{u}} f(a,b)$ gives the change in f in the $\mathbf{u}$ direction.
In our situtation $D_{\mathbf{u}} f(1,2) = |\nabla f(1,2)| = \sqrt{41}$.

Answer: I will have to climb approximately $\sqrt{41}$ units.