Name ______________________________ Math 291, 01, Exam #1  Feb., 2006

**Be sure to show all your work. Unsupported answers will receive no credit.**

**A formula sheet is supplied, for your reference.**

**Use the backs of the exam pages for scratchwork or for continuation of your answers, if necessary.**

| **Problem No.** | **Pts Possible** | **Points** |
|---|---|---|
| 1 | 12 | |
| 2 | 13 | |
| 3 | 14 | |
| 4 | 14 | |
| 5 | 14 | |
| 6 | 20 | |
| 7 | 13 | |
| | | |
| **Total** | 100 | |

**1. (12 points):** "Thank you Dr. Cook! This problem was a gift."

(a) Consider the points $(1, 2, 3)$, $(4, 5, 6)$, and $(1, 1, 1)$. Find the equation of the plane which contains these points.

(b) Find parametric equations for a line which is perpendicular to the plane found in part (a) and which intersects that plane at the point $(1, 2, 3)$.

(c) Find the area of the parallelogram with vertices: $(1, 2, 3)$, $(4, 5, 6)$, $(1, 1, 1)$, and $(4, 4, 4)$.

**2. (13 points):** Choose **one** of the following:

I. Given a vector $\mathbf{a} \neq \mathbf{0}$ such that $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c}$ and $\mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{c}$, show that we must have $\mathbf{b} = \mathbf{c}$.

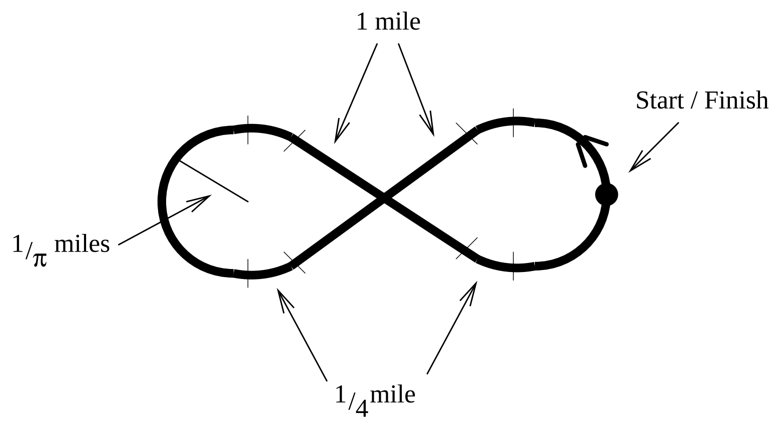
II. Given that $\mathbf{r}(t) \neq 0$ and $\mathbf{r}(t)$ is differentiable, show that

$$\frac{d}{dt}|\mathbf{r}(t)| = \frac{1}{|\mathbf{r}(t)|}\mathbf{r}(t) \cdot \mathbf{r}'(t).$$

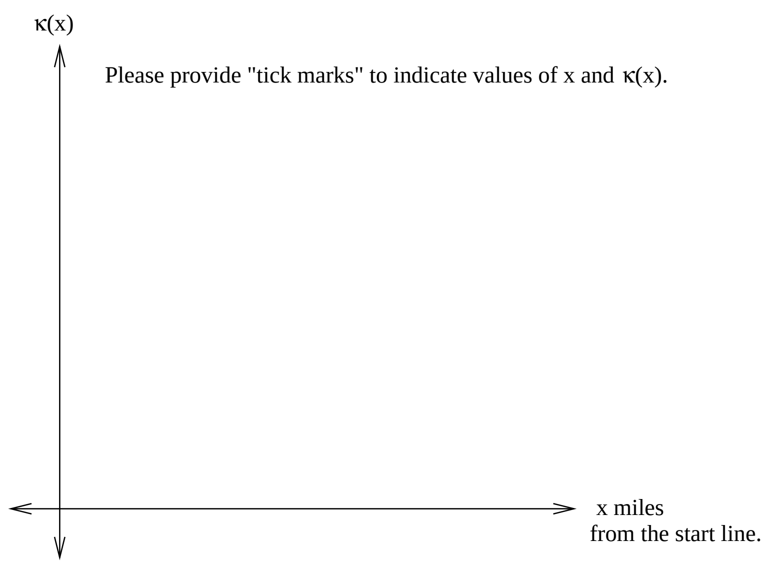
**3. (14 points):** Some differential geometry.

(a) I am a curve with a smooth parametrization $\mathbf{r}(t)$. If you compute my $\mathbf{B}(t)$ vector, you will find that it is a (well-defined, non-zero) constant vector for all $t$. My curvature, $\kappa(t)$, is constant for all $t$. What am I?

(b) Here is a diagram of "Bill's Less Than Safe Speedway" (it's not exactly safe because it crosses itself in the middle).




Use this diagram, to sketch the speedway's curvature on the axes below.




(c) Give your best guess as to what $\tau(t)$ should be for my speedway. And (as always) explain your choice.

**4. (14 points):** Continuity $\neq$ Fun. Proof: Given any $\epsilon > 0$ there exists pain $> 0$...

(a) Can you find a number $A$ which makes

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & (x,y) \neq (0,0) \\ A & (x,y) = (0,0) \end{cases}$$

continuous everywhere?

If not, explain why. If so, find $A$ and show that $f$ is continous everywhere.

(b) Can you find a number $A$ which makes

$$f(x,y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}} & (x,y) \neq (0,0) \\ A & (x,y) = (0,0) \end{cases}$$

continuous everywhere?

If not, explain why. If so, find $A$ and show that $f$ is continous everywhere.

**5. (14 points):** Choose **one** of the following:

I. Given $z = f(x, y)$ is a differentiable function, $x(s,t) = s^2+t^2$, and $y(s,t) = s^2-t^2$ show that

$$\frac{\partial z}{\partial s}\frac{\partial z}{\partial t} = 4st\left(\left(\frac{\partial z}{\partial x}\right)^2 - \left(\frac{\partial z}{\partial y}\right)^2\right)$$

II. State and prove the chain rule for $z = f(x, y)$, $x = g(t)$, and $y = h(t)$ where $f$, $g$, and $h$ are differentiable everywhere.

**6. (20 points):** An odd collection of problems. Let $F(x, y, z) = xy - z^2$.

(a) Find the equation of the plane tangent to $F(x, y, z) = 1$ at $(2, 1, -1)$.

(b) Compute $\dfrac{\partial z}{\partial x}$ given $xy - z^2 = 1$.

(c) What point(s) of the surface $F(x, y, z) = 1$ are closest to the origin?

**7. (13 points):** Loose ends.

(a) Find and classify all critical points of $z = -x^3 + 4xy - 2y^2 + 1$.

(b) I am standing on the surface $z = -x^3 + 4xy - 2y^2 + 1$ at the point $(1, 2, 0)$. I want to climb up hill in the fastest possible manner. What direction should I face before I start climbing (give your answer in the form of a **unit** vector $\mathbf{u} = \langle a, b \rangle$)?

(c) In order to move 1 unit horizontally in the direction found in part (b), approximately how far up will I have to climb?

# Math 291 Spring 2006: Formulas for Exam #1

Distance between $(x_0, y_0, z_0)$ and $(x_1, y_1, z_1)$ is: $\sqrt{(x_0 - x_1)^2 + (y_0 - y_1)^2 + (z_0 - z_1)^2}$.

Distance between $(x_1, y_1, z_1)$ and $ax + by + cz + d = 0$ is: $\dfrac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$.

A line with direction $\mathbf{v}$ through a point $\mathbf{r}_0$ is given by: $\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}$.

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos(\theta) \quad \text{and} \quad |\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin(\theta)$$

$$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}, \quad \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}, \quad \text{and} \quad \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$$

The area of a parallelogram spanned by $\mathbf{a}$ and $\mathbf{b}$ is $|\mathbf{a} \times \mathbf{b}|$.
The volume of a parallelepiped spanned by $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ is $|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$.

$$\text{comp}_{\mathbf{a}}(\mathbf{b}) = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|} \quad \text{and} \quad \text{proj}_{\mathbf{a}}(\mathbf{b}) = \frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{a} \cdot \mathbf{a}}\mathbf{a}$$

The linearization of (equation of the tangent plane to) $z = f(x, y)$ at $(x_0, y_0, f(x_0, y_0))$:

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

If $z = f(x_1, x_2, \ldots, x_n)$, then $dz = f_{x_1}dx_1 + f_{x_2}dx_2 + \cdots + f_{x_n}dx_n$.
The tangent plane to $F(x, y, z) = K$ at $(a, b, c)$ is: $\nabla F(a, b, c) \cdot \langle x - a, y - b, z - c\rangle = 0$.

$$\nabla f(x, y, z) = \langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z)\rangle \text{ and } D_{\mathbf{u}}f(x, y, z) = \nabla f(x, y, z) \cdot \mathbf{u}.$$

If $y = f(x)$ is given implicitly by $F(x, y) = K$, then $\dfrac{dy}{dx} = -\dfrac{F_x}{F_y}$.

If $z = f(x, y)$ is given implicitly by $F(x, y, z) = K$, then $\dfrac{\partial z}{\partial x} = -\dfrac{F_x}{F_z}$ and $\dfrac{\partial z}{\partial y} = -\dfrac{F_y}{F_z}$.

Arc length from $a$ to $t$ is given by the function $s(t) = \displaystyle\int_a^t |\mathbf{r}'(u)|\, du$. Thus $s'(t) = |\mathbf{r}'(t)|$.

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} \qquad \mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} \qquad \mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$$

$$\mathbf{T}'(t) = \kappa(t)s'(t)\mathbf{N}(t) \qquad \mathbf{N}'(t) = -\kappa(t)s'(t)\mathbf{T}(t) + \tau(t)s'(t)\mathbf{B}(t) \qquad \mathbf{B}'(t) = -\tau(t)s'(t)\mathbf{N}(t)$$

$\kappa(t) = \left|\dfrac{d\mathbf{T}}{ds}\right| = \dfrac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \dfrac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$ For the curve $y = f(x)$: $\kappa(x) = \dfrac{|f''(x)|}{(1 + (f'(x))^2)^{3/2}}$.

$$a_T = \frac{\mathbf{r}'(t) \cdot \mathbf{r}''(t)}{|\mathbf{r}'(t)|} \qquad a_N = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|} \qquad \tau(t) = \frac{(\mathbf{r}'(t) \times \mathbf{r}''(t)) \cdot \mathbf{r}'''(t)}{|\mathbf{r}'(t) \times \mathbf{r}''(t)|^2}$$

For $z = f(x, y)$, define: $D(a, b) = \begin{vmatrix} f_{xx}(a, b) & f_{xy}(a, b) \\ f_{yx}(a, b) & f_{yy}(a, b) \end{vmatrix} = f_{xx}(a, b)f_{yy}(a, b) - f_{xy}(a, b)^2$.

Let $(a, b)$ be a critical point for $z = f(a, b)$. Then,

- If $D(a, b) > 0$ and $f_{xx}(a, b) > 0$, then $z = f(x, y)$ has a relative minimum at $(a, b)$.
- If $D(a, b) > 0$ and $f_{xx}(a, b) < 0$, then $z = f(x, y)$ has a relative maximum at $(a, b)$.
- If $D(a, b) < 0$, then $z = f(x, y)$ has a saddle point at $(a, b)$.