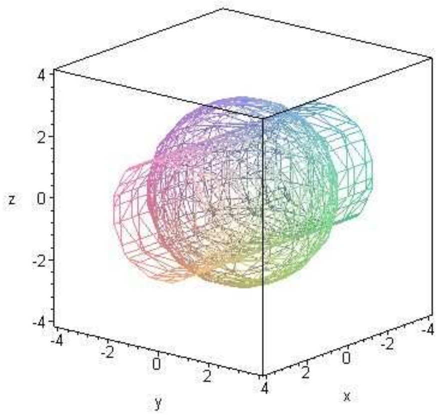
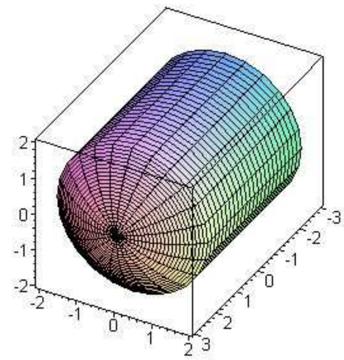
Math 291 Spring 2006
Exam #2: Answer Key

1. (10pts) Find the volume of the solid inside both $x^2 + y^2 + z^2 = 9$ and $y^2 + z^2 = 4$.





We are finding the volume of a solid which lies inside a sphere ($x^2 + y^2 + z^2 = 9$) and a cylinder ($y^2 + z^2 = 4$). Since the cylinder is symmetric with respect to the x-axis. Let's solve the sphere's equation for x. We get: $x = \pm\sqrt{9 - y^2 - z^2}$. This gives the "top" and the "bottom" of our solid. Now the cylinder comes into play. It tells us that we should integrate over the whole region $y^2 + z^2 \leq 9$ (this would give us the volume of the whole spere). Instead we should integrate over the region $y^2 + z^2 \leq 4$ (that is inside the cylinder). So we get the following triple integral:

$$\iint_{y^2+z^2\leq 4} \int_{-\sqrt{9-y^2-z^2}}^{\sqrt{9-y^2-z^2}} 1\, dx\, dA$$

Since we are going to integrate over the inside of a circle, let's switch to some kind of polar coordinates (i.e. $y = r\cos(\theta)$ and $z = r\sin(\theta)$). Then $y^2 + z^2 = r^2$ and our integral changes to:

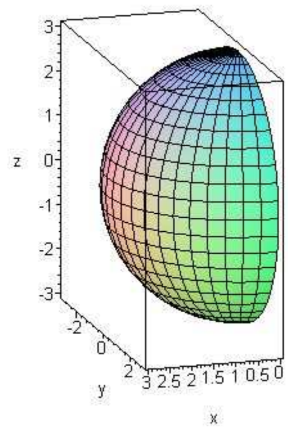
$$\begin{aligned}
\int_0^{2\pi} \int_0^2 \int_{-\sqrt{9-r^2}}^{\sqrt{9-r^2}} r\, dx\, dr\, d\theta &= \int_0^{2\pi} \int_0^2 r\left(\sqrt{9-r^2} - (-\sqrt{9-r^2})\right) dr\, d\theta \\
&= \int_0^{2\pi} \int_0^2 2r\sqrt{9-r^2}\, dr\, d\theta \\
&= \int_0^{2\pi} -\frac{2}{3}(9-r^2)^{3/2}\Big|_0^2\, d\theta \\
&= \int_0^{2\pi} -\frac{2}{3}(9-2^2)^{3/2} + \frac{2}{3}(9-0^2)^{3/2}\, d\theta \\
&= \int_0^{2\pi} 18 - \frac{10\sqrt{5}}{3}\, d\theta \\
&= 36\pi - \frac{20\sqrt{5}}{3}\pi
\end{aligned}$$

2. (12pts) Evaluate the following triple integral:

$$\int_0^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_{-\sqrt{9-x^2-y^2}}^{\sqrt{9-x^2-y^2}} \frac{1}{\sqrt{x^2+y^2+z^2}}\, dz\, dy\, dx$$

We are obviously integrating over part of a sphere centered at the origin of radius 3, so switching to spherical coordinates makes sense.

Looking at the z and y bounds, we have both "top" and "bottom" also "left" and "right" sides of the sphere represented. However, the x bounds go from 0 to 3 instead of -3 to 3. So the "front" of the sphere is included, but not the "back". Therefore, ρ should range from 0 to 3, ϕ should range from 0 to π, but θ should sweep from $-\pi/2$ to $\pi/2$.



Also, notice that $\frac{1}{\sqrt{x^2+y^2+z^2}} = \frac{1}{\sqrt{\rho^2}}$ (and don't forget the Jacobian). Thus we get:

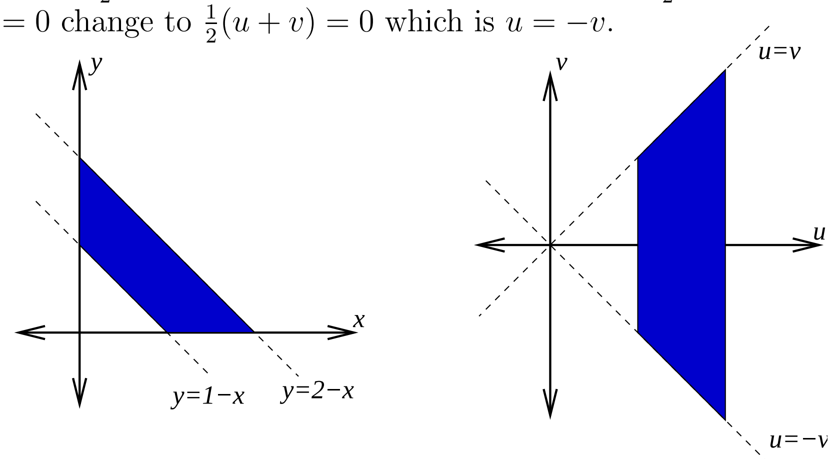
$$\begin{aligned}
\int_0^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_{-\sqrt{9-x^2-y^2}}^{\sqrt{9-x^2-y^2}} \frac{1}{\sqrt{x^2+y^2+z^2}}\, dz\, dy\, dx \\
&= \int_{-\pi/2}^{\pi/2} \int_0^{\pi} \int_0^3 \frac{1}{\rho}\rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta \\
&= \int_{-\pi/2}^{\pi/2} d\theta \int_0^{\pi} \sin(\phi)\, d\phi \int_0^3 \rho\, d\rho \\
&= \pi \cdot 2 \frac{9}{2} = 9\pi
\end{aligned}$$

3. (15pts) Let R be the tapezoidal region with vertices $(1,0)$, $(2,0)$, $(0,1)$, and $(0,2)$. Evaluate the following integral:

$$\iint_R \cos\left(\frac{y-x}{y+x}\right)\, dA$$

Hint: Pick a change of variables which simplifies the argument of the cosine. First, let's find the equations of the lines which define the edges of our region. The segment from $(1,0)$ to $(2,0)$ is part of the line $y=0$. The segment from $(0,1)$ to $(0,2)$ is part of the line $x=0$. The segment from $(0,1)$ to $(1,0)$ is part of the line $y=-x+1$. The segment from $(0,2)$ to $(2,0)$ is part of the line $y=-x+2$.

Integrating $\cos\left(\frac{y-x}{y+x}\right)$ is too difficult as it stands, so we try a change of variables. Choosing $u=y-x$ and $v=y+x$ would simplify things. Let's see how our bounds change. $y=-x+1$ changes to $v=1$ and $y=-x+2$ changes to $v=2$. Next, solving our transformation equations for x and y, we get that $u+v=2y$ and $v-u=2x$. Thus $x=\frac{1}{2}(v-u)$ and $y=\frac{1}{2}(u+v)$. Thus the line $x=0$ changes to $\frac{1}{2}(v-u)=0$ which is $u=v$. And the line $y=0$ change to $\frac{1}{2}(u+v)=0$ which is $u=-v$.



Finally, we compute the Jacobian of this transformation and get:

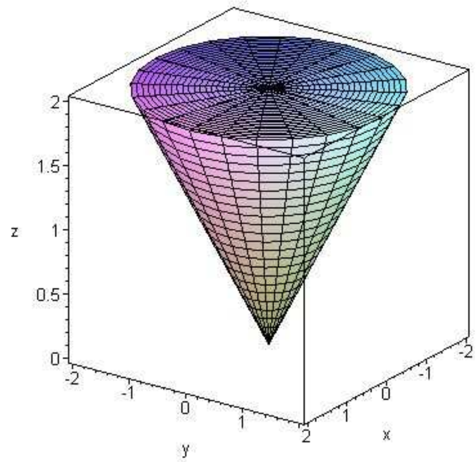
$$\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \det\begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2}$$

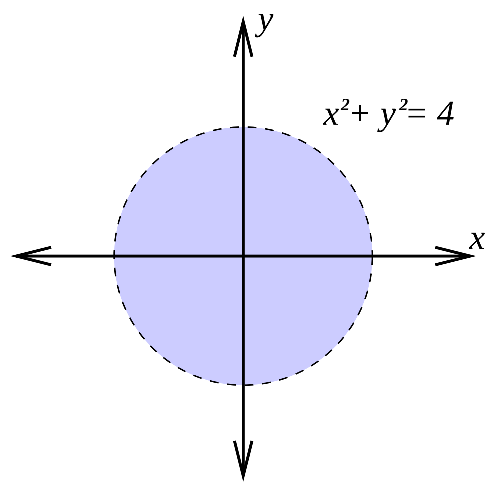
Thus we get that:

$$\begin{aligned}
\iint_R \cos\left(\frac{y-x}{y+x}\right) dA &= \int_1^2 \int_{-v}^{v} \cos\left(\frac{u}{v}\right) \left|-\frac{1}{2}\right| du\, dv \\
&= \int_1^2 \frac{v}{2} \sin\left(\frac{u}{v}\right)\Big|_{-v}^{v}\, dv \\
&= \int_1^2 \frac{v}{2}\sin\left(\frac{v}{v}\right) - \frac{v}{2}\sin\left(\frac{-v}{v}\right) dv \\
&= \int_1^2 v\sin(1)\, dv \\
&= \frac{v^2}{2}\sin(1)\Big|_1^2 \\
&= \frac{4}{2}\sin(1) - \frac{1}{2}\sin(1) = \frac{3}{2}\sin(1)
\end{aligned}$$

4. (13pts) Consider,

$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{2} f(x,y,z)\,dz\,dy\,dx.$$





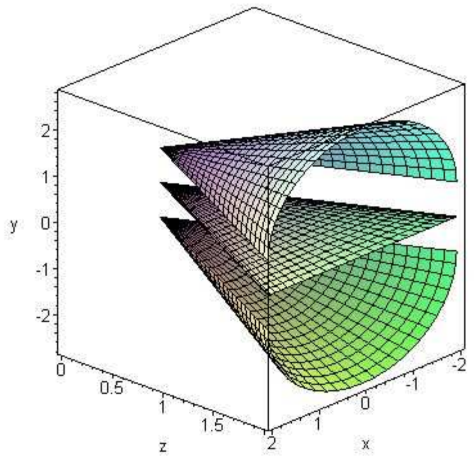
(a) Change the order of integration, so z is first, x is second, and y is last. Just focusing on x's and y's bounds, we see that we are integrating over a circle centered at the origin of radius 2. Therefore, solving $x^2 + y^2 = 4$ for x instead of y we get our desired bounds.

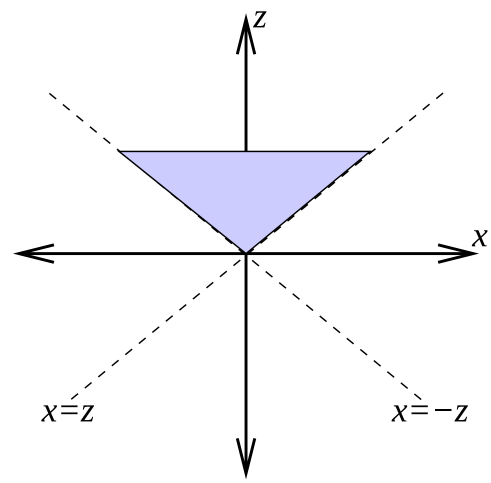
$$\int_{-2}^{2}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}}\int_{\sqrt{x^2+y^2}}^{2} f(x,y,z)\,dz\,dx\,dy$$

(b) Rewrite this integral in cylindrical coordinates. This just involves changes the x, y bounds to r, θ bounds. From part (a), we know that x and y vary over the interior of the circle $x^2+y^2 = 4$. Thus r should range from 0 to 2 and θ should range from 0 to 2π. The z bound $z = 2$ doesn't need to be changed. But $z = \sqrt{x^2 + y^2} = \sqrt{r^2} = r$. Thus we get (don't forget the Jacobian!):

$$\int_{0}^{2\pi}\int_{0}^{2}\int_{r}^{2} f(r\cos(\theta), r\sin(\theta), z)\,r\,dz\,dr\,d\theta$$

(c) Change the order of integration, so y is first, x is second, and z is last. Now this is a little harder. From part (a), we know that x and y are varying over the interior of $x^2+y^2 = 4$. z is bounded above by the plane $z = 2$ and below by the surface $z = \sqrt{x^2 + y^2}$ which happens to be a cone. So our entire region of integration is just the inside of the part of the cone $z = \sqrt{x^2 + y^2}$ whose z coordinates are between 0 and 2.





We want y's bounds to come first, so we solve $z = \sqrt{x^2+y^2}$ for y and get $y = \pm\sqrt{z^2-x^2}$. Let's intersect the cone's equation with the xz-plane (i.e. $y = 0$) and see what we get: $0 = \pm\sqrt{z^2-x^2}$. This means that $z^2 = x^2$. Which is $x = \pm z$.

Finally, z ranges from 0 to 2. We have the following:

$$\int_0^2 \int_{-z}^{z} \int_{-\sqrt{z^2-x^2}}^{\sqrt{z^2-x^2}} f(x,y,z)\, dy\, dx\, dz$$

5. (15pts) A few odds and ends. Let f, g, and h be smooth functions.

(a) Determine if $\mathbf{F}(x,y,z) = (y\ln(z) - y\sin(xy))\,\mathbf{i} + (x\ln(z) - x\sin(xy))\,\mathbf{j} + \left(\frac{xy}{z} + 2z\right)\mathbf{k}$ is a conservative vector field. Let's check to see if $\text{curl}(\mathbf{F}) = \mathbf{0}$ (if so, then $\mathbf{F}$ is conservative).

$$\text{curl}(\mathbf{F}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (y\ln(z) - y\sin(xy)) & (x\ln(z) - x\sin(xy)) & \left(\frac{xy}{z} + 2z\right) \end{vmatrix}$$

$$= \langle \frac{x}{z} - \frac{x}{z}, -\left(\frac{y}{z} - \frac{y}{z}\right), (\ln(z) - \sin(xy) - xy\cos(xy)) - (\ln(z) - \sin(xy) - xy\cos(xy)), \rangle$$

Therefore, $\text{curl}(\mathbf{F}) = \mathbf{0}$. Thus $\mathbf{F}$ is conservative.

Another method for determining if F is conservative, is to try to construct a potential function. We get that

$$f(x,y,z) = \int y\ln(z) - y\sin(xy)\, dx = xy\ln(z) + \cos(xy) + C_1(y,z)$$

$$f(x,y,z) = \int x\ln(z) - x\sin(xy)\, dy = xy\ln(z) + \cos(xy) + C_1(x,z)$$

$$f(x,y,z) = \int \frac{xy}{z} + 2z\, dz = xy\ln(z) + z^2 + C_1(x,y)$$

Putting this together we find that $f(x,y,z) = xy\ln(z) + \cos(xy) + z^2 + C$ (any constant C). Since $\nabla f = \mathbf{F}$, we have that $\mathbf{F}$ is conservative.

(b) Determine if $\mathbf{F}(x,y,z) = f(x)\mathbf{i} + g(y)\mathbf{j} + h(z)\mathbf{k}$ is a conservative vector field.

$$\text{curl}(\mathbf{F}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f(x) & g(y) & h(z) \end{vmatrix} = \langle 0-0, 0-0, 0-0 \rangle = \mathbf{0}$$

Therefore, $\mathbf{F}$ is conservative.

Again we could try to construct a potential function. Here the function is $\int f(x)\,dx + \int g(y)\,dy + \int h(z)\,dz + C$.

(c) Prove that $\text{div}\,(\nabla f \times \nabla g) = 0$. We compute $\nabla f \times \nabla g$ first.

$$\nabla f \times \nabla g = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} & \frac{\partial g}{\partial z} \end{vmatrix} = \langle f_y g_z - f_z g_y, -(f_x g_z - f_z g_x), f_x g_y - f_y g_x \rangle$$

Therefore,

$$\begin{aligned} &\text{div}\,(\nabla f \times \nabla g) \\ =\ & \text{div}\,(\langle f_y g_z - f_z g_y, -(f_x g_z - f_z g_x), f_x g_y - f_y g_x \rangle) \\ =\ & \frac{\partial}{\partial x}(f_y g_z - f_z g_y) - \frac{\partial}{\partial y}(f_x g_z - f_z g_x) + \frac{\partial}{\partial z}(f_x g_y - f_y g_x) \\ =\ & f_{yx} g_z + f_y g_{zx} - f_{zx} g_y - f_z g_{yx} - f_{xy} g_z - f_x g_{zy} \\ & + f_{zy} g_x + f_z g_{xy} + f_{xz} g_y + f_x g_{yz} - f_{yz} g_x - f_y g_{xz} \\ =\ & (f_{yx} - f_{xy}) g_z + f_y (g_{zx} - g_{xz}) + (f_{xz} - f_{zx}) g_y \\ & + f_z (g_{xy} - g_{yx}) + f_x (g_{yz} - g_{zy}) + (f_{zy} - f_{yz}) g_x \quad = \quad 0 \end{aligned}$$

(Use Clairaut's Theorem repeatedly.)

6. (15pts) Compute the following line integrals.

(a) $\displaystyle\int_C \frac{e^y}{x}\,dz$ where C is parametrized by $\mathbf{r}(t) = \langle t, t, t^2 \rangle$ and $0 \le t \le 1$.

The vector field in question is not conservative and C is not a closed curve, so we will compute this without any special tricks.

$dz = z'(t)\,dt = 2t\,dt$ so we get that:

$$\int_C \frac{e^y}{x}\,dz = \int_0^1 \frac{e^t}{t} 2t\,dt = \int_0^1 2e^t\,dt = 2e - 2$$

(b) $\displaystyle\int_C ye^{xy}\,dx + xe^{xy}\,dy$ where C is the arc $y = x^2$ from $(0,0)$ to $(1,1)$.

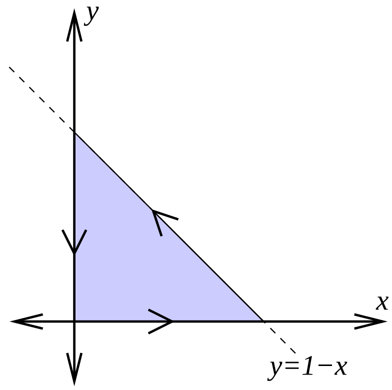
Notice that

$$\int ye^{xy}\,dx = e^{xy} + C = \int xe^{xy}\,dy$$

so if we choose $f(x,y) = e^{xy}$, then $\nabla f(x,y) = \langle xe^{xy}, ye^{xy} \rangle$. So by the fundamental theorem of line integrals:

$$\int_C ye^{xy}\,dx + xe^{xy}\,dy = f(1,1) - f(0,0) = e^{(1)(1)} - e^{(0)(0)} = e - 1$$

(c) $\int_C xy\,dx + xy\,dy$ where C is the edges of the triangle with vertices $(0,0)$, $(1,0)$, and $(0,1)$ oriented counter-clockwise.

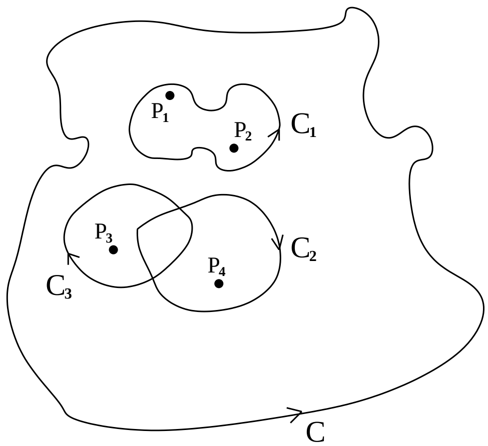


We are integrating around a positively oriented simple closed curve C, so let's use Green's Theorem (we could compute this integral straight from the definition, but it would be too much work).

The region bounded by C is bounded above by $y = -x + 1$ and bounded below by $y = 0$ as x ranges from 0 to 1. Thus we get that:

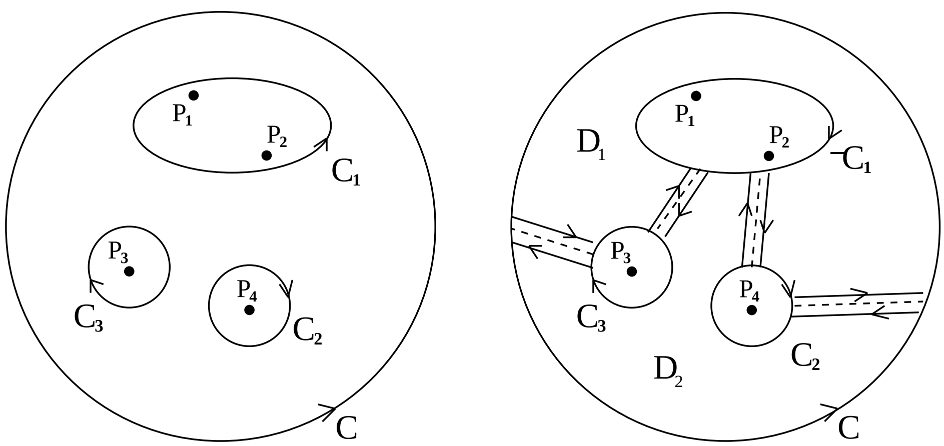
$$\begin{aligned}
\int_C xy\,dx + xy\,dy &= \int_0^1 \int_0^{-x+1} \frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial y}(xy)\;dy\,dx \\
&= \int_0^1 \int_0^{-x+1} y - x\,dy\,dx \\
&= \int_0^1 \frac{1}{2}y^2 - xy\Big|_0^{-x+1}\,dx \\
&= \int_0^1 \frac{1}{2}(-x+1)^2 - x(-x+1)\,dx \\
&= \int_0^1 \frac{3}{2}x^2 - 2x + \frac{1}{2}\,dx \\
&= \frac{1}{2}x^3 - x^2 + \frac{1}{2}x\Big|_0^1 \\
&= \frac{1}{2} - 1 + \frac{1}{2} = 0
\end{aligned}$$

7. (10pts) Let $\mathrm{F}(x, y) = \langle P(x, y), Q(x, y)\rangle$ be a vector field defined on all of $\mathbb{R}^2$ except the points P_1, P_2, P_3 and P_4. In addition assume that the first partial derivatives of P and Q exist and are continuous (except at those troublesome points). Finally, assume that $\mathbf{F}$ is conservative everywhere it is defined.



Given that $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = 2$, $\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = -1$, and $\int_{C_3} \mathbf{F} \cdot d\mathbf{r} = 5$, find $\int_{C} \mathbf{F} \cdot d\mathbf{r}$.

Recall that any deformation of a curve in $\mathbb{R}^2$ (which doesn't cross a "bad spot" of our vector field) leaves a line integral's value unchanged. Thus we can deform the curves C_1, C_2, C_3, and C so that we have:



We reverse the orientation of C_1 and make a few harmless "cuts". As usual, we notice that the integrals along the cuts cancel each other out. Putting all of these together (and using Green's Theorem) we see that:

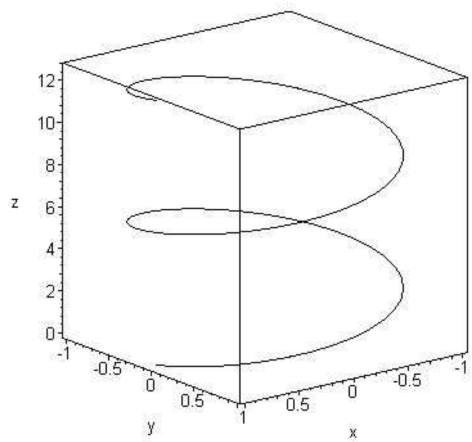
$$\int_{C - C_1 + C_2 + C_3} \mathbf{F} \cdot d\mathbf{r} = \iint_{D_1} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA + \iint_{D_2} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA = 0$$

The right-hand side is zero, because $\dfrac{\partial Q}{\partial x} = \dfrac{\partial P}{\partial y}$ since $\mathbf{F}$ is conservative (away from the bad spots).

Now we just spit up the integral on the left-hand side and solve for $\int_C$. We get:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} - \int_{C_2} \mathbf{F} \cdot d\mathbf{r} - \int_{C_3} \mathbf{F} \cdot d\mathbf{r} = 2 - (-1) - 5 = -2$$

8. (10pts) A thin wire with constant density ρ is bent into a helix whose shape is given by $\mathbf{r}(t) = \langle \cos(t), \sin(t), t \rangle$ where $0 \le t \le 4\pi$. Find the bent wire's the center of mass.



Let's calculate ds first. $ds = |\mathbf{r}'(t)|\,dt = |\langle -\sin(t), \cos(t), 1\rangle|\,dt$
$= \sqrt{\sin^2(t) + \cos^2(t) + 1}\,dt = \sqrt{2}\,dt.$

Thus we have that

$$m = \int_C \rho\,ds = \int_0^{4\pi} \rho\sqrt{2}\,dt = 4\sqrt{2}\pi\rho$$

$$M_{yz} = \int_C \rho x\,ds = \int_0^{4\pi} \rho\cos(t)\sqrt{2}\,dt = 0$$

$$M_{xz} = \int_C \rho y\,ds = \int_0^{4\pi} \rho\sin(t)\sqrt{2}\,dt = 0$$

$$M_{xy} = \int_C \rho z\,ds = \int_0^{4\pi} \rho t\sqrt{2}\,dt = \rho\sqrt{2}\frac{t^2}{2}\Big|_0^{4\pi} = \rho\sqrt{2}\frac{16\pi^2}{2} = \rho 8\sqrt{2}\pi^2$$

Therefore,

$$(\bar{x}, \bar{y}, \bar{z}) = \frac{1}{m}\,(M_{yz}, M_{xz}, M_{yz}) = (0, 0, 2\pi)$$