Name ______________________________ Math 291, 01, Exam #2 Apr., 2006

Be sure to show all your work. Unsupported answers will receive no credit.

A formula sheet is supplied, for your reference.

Use the backs of the exam pages for scratchwork or for continuation of your answers, if necessary.

Problem No.	**Pts Possible**	**Points**
1	10	
2	12	
3	15	
4	13	
5	15	
6	15	
7	10	
8	10	
Total	100	

1. (10pts): Find the volume of the solid inside both $x^2 + y^2 + z^2 = 9$ and $y^2 + z^2 = 4$.

2. (12pts): Evaluate the following triple integral:

$$\int_0^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_{-\sqrt{9-x^2-y^2}}^{\sqrt{9-x^2-y^2}} \frac{1}{\sqrt{x^2+y^2+z^2}}\, dz\, dy\, dx$$

3. (15pts): Let R be the tapezoidal region with vertices $(1,0)$, $(2,0)$, $(0,1)$, and $(0,2)$. Evaluate the following integral:

$$\iint_R \cos\left(\frac{y-x}{y+x}\right)\, dA$$

Hint: Pick a change of variables which simplifies the argument of the cosine.

4. (13pts): Consider,

$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{2} f(x,y,z)\,dz\,dy\,dx.$$

(a) Change the order of integration, so z is first, x is second, and y is last.

(b) Rewrite this integral in cylindrical coordinates.

(c) Change the order of integration, so y is first, x is second, and z is last.

5. (15pts): A few odds and ends. Let f, g, and h be smooth functions.

(a) Determine if $\mathbf{F}(x, y, z) = (y \ln(z) - y \sin(xy))\, \mathbf{i} + (x \ln(z) - x \sin(xy))\, \mathbf{j} + \left(\frac{xy}{z} + 2z\right) \mathbf{k}$ is a conservative vector field.

(b) Determine if $\mathbf{F}(x, y, z) = f(x)\mathbf{i} + g(y)\mathbf{j} + h(z)\mathbf{k}$ is a conservative vector field.

(c) Prove that $\operatorname{div}(\nabla f \times \nabla g) = 0$.

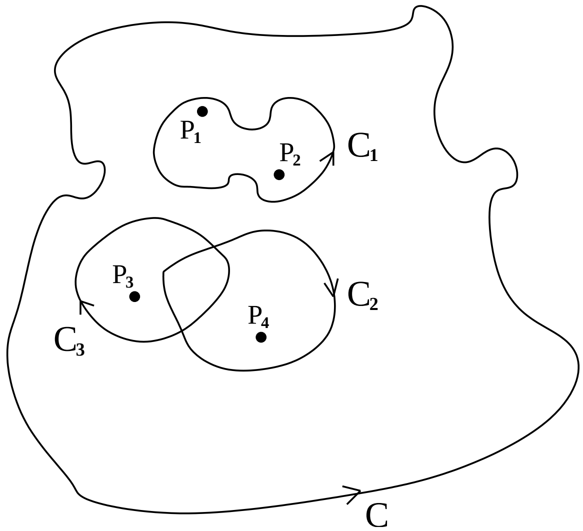
6. (15pts): Compute the following line integrals.

(a) $\displaystyle\int_C \frac{e^y}{x}\,dz$ where C is parametrized by $\mathbf{r}(t) = \langle t, t, t^2 \rangle$ and $0 \leq t \leq 1$.

(b) $\displaystyle\int_C ye^{xy}\,dx + xe^{xy}\,dy$ where C is the arc $y = x^2$ from $(0,0)$ to $(1,1)$.

(c) $\displaystyle\int_C xy\,dx + xy\,dy$ where C is the edges of the triangle with vertices $(0,0)$, $(1,0)$, and $(0,1)$ oriented counter-clockwise.

7. (10pts): Let $\mathrm{F}(x, y) = \langle P(x, y), Q(x, y)\rangle$ be a vector field defined on all of $\mathbb{R}^2$ except the points P_1, P_2, P_3 and P_4. In addition assume that the first partial derivatives of P and Q exist and are continuous (except at those troublesome points). Finally, assume that $\mathbf{F}$ is conservative everywhere it is defined.



Given that $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = 2$, $\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = -1$, and $\int_{C_3} \mathbf{F} \cdot d\mathbf{r} = 5$, find $\int_{C} \mathbf{F} \cdot d\mathbf{r}$.

8. (10pts): A thin wire with constant density ρ is bent into a helix whose shape is given by $\mathbf{r}(t) = \langle \cos(t), \sin(t), t \rangle$ where $0 \leq t \leq 4\pi$. Find the bent wire's the center of mass.

Math 291 Spring 2006: Formulas for Exam #2

$$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos(\theta) \quad \text{and} \quad |\mathbf{a}\times\mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin(\theta)$$

The area of a parallelogram spanned by $\mathbf{a}$ and $\mathbf{b}$ is $|\mathbf{a}\times\mathbf{b}|$.
The volume of a parallelepiped spanned by $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ is $|\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})|$.

The tangent plane to $F(x,y,z)=K$ at (a,b,c) is: $\nabla F(a,b,c)\cdot\langle x-a, y-b, z-c\rangle = 0$.

$$\nabla f(x,y,z) = \langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z)\rangle \text{ and } D_{\mathbf{u}}f(x,y,z) = \nabla f(x,y,z)\cdot\mathbf{u}.$$

Arc length from a to t is given by the function $s(t) = \displaystyle\int_a^t |\mathbf{r}'(u)|\,du$. Thus $s'(t) = |\mathbf{r}'(t)|$.

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} \qquad \mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} \qquad \kappa(t) = \frac{d\mathbf{T}}{ds} = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$$

Suppose that a large object of mass M is located at the origin and an object of mass m is located at the point $\langle x,y,z\rangle$. Then the gravatational force of M on m is $\mathbf{F}(x,y,z) = -\frac{mMG}{\sqrt{x^2+y^2+z^2}}\langle x,y,z\rangle$ where G is the gravatational constant.

Center of mass - 3 dimensions

$$\text{mass} = m = \iiint_E \rho(x,y,z)\,dV \qquad (\bar{x},\bar{y},\bar{z}) = \frac{1}{m}(M_{yz}, M_{xz}, M_{xy})$$

$$M_{yz} = \iiint_E x\,\rho(x,y,z)\,dV, \qquad M_{xz} = \iiint_E y\,\rho(x,y,z)\,dV, \qquad M_{xy} = \iiint_E z\,\rho(x,y,z)\,dV$$

Center of mass - a wire in the plane

$$\text{mass} = m = \int_C \rho(x,y)\,ds \qquad (\bar{x},\bar{y}) = \frac{1}{m}(M_y, M_x)$$

$$M_y = \int_C x\,\rho(x,y)\,ds, \qquad M_x = \int_C y\,\rho(x,y)\,ds$$

Polar: $x = r\cos(\theta)$, $y = r\sin(\theta)$, $dA = r\,drd\theta$

Cylindrical: $x = r\cos(\theta)$, $y = r\sin(\theta)$, $z = z$, $dV = r\,drd\theta dz$

Spherical: $x = \rho\sin(\phi)\cos(\theta)$, $y = \rho\sin(\phi)\sin(\theta)$, $z = \rho\cos(\phi)$, $dV = \rho^2\sin(\phi)\,d\rho d\theta d\phi$

Change of variables:

$$\iint_R f(x,y)\,dx\,dy = \iint_S f(x(u,v),y(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\,du\,dv$$

$$\iiint_R f(x,y,z)\,dx\,dy\,dz = \iiint_S f(x(u,v,w),y(u,v,w),z(u,v,w))\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right|\,du\,dv\,dw$$

$$ds = |\mathbf{r}'(t)|dt \qquad dx = x'(t)dt \qquad \int_C \mathbf{F}\cdot d\mathbf{r} = \int_C \mathbf{F}\cdot\mathbf{T}\,ds$$

Green's Theorem: $\displaystyle\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\,dA = \iint_D \text{curl}(\mathbf{F})\cdot\mathbf{k}\,dA$