Math 3110 **Final Exam** **December 9^{th}, 2009**

Name: ______________________________

Don't merely state answers, prove your statements. **Be sure to show your work!**

The last page is a copy of Cayley tables for D_4 and Q.

1. (____/12 points) For each of the following pairs of groups, if the groups are isomorphic, circle $G_1 \cong G_2$ and explain why they are ismorphic. If the groups aren't isomorphic, circle $G_1 \ncong G_2$ and explain why not.

(a) $\mathbb{Q} \cong Q$ OR $\mathbb{Q} \ncong Q$ [$\mathbb{Q}$ = rationals, Q = quaternions]

(b) $A_4 \cong D_6$ OR $A_4 \ncong D_6$ [A_4 = even permutations in S_4]

(c) $U(5) \cong \mathbb{Z}_4$ OR $U(5) \ncong \mathbb{Z}_4$

2. (____/12 points) Sub-Things

(a) Let G be a group. Recall that $Z(G) = \{x \in G \,|\, gx = xg \text{ for all } g \in G\}$ is the "center" of G. Prove that $Z(G)$ is a normal subgroup of G. [Hint: You may either *show it is a subgroup* **and** *show it is normal*, or you can find a homomorphism with kernel $Z(G)$.]

(b) Let $S = \{m + n\sqrt{5} \,|\, m, n \in \mathbb{Z}\}$. Show S is a sub**ring** of $\mathbb{R}$.

(c) Let $T = \{m + n\sqrt{5} \,|\, m, n \in \mathbb{Z} \text{ and } m \text{ is } \textbf{EVEN}\}$. Show T is a **subgroup** of $\mathbb{R}$ (under addition of course) and then give a concrete counter-example which shows why T is **not** a sub**ring** of $\mathbb{R}$.

3. (____/13 points) Calculatin' mod 50. [$\mathbf{50 = 2 \cdot 5^2}$]

(a) Is 3 a unit, zero divisor, or neither in $\mathbb{Z}_{50}$? If 3 is a unit, find its inverse. If 3 is zero divisor, show this by finding some $0 \neq m \in \mathbb{Z}_{50}$ such that $3m = 0 \pmod{50}$.

(b) Is 20 a unit, zero divisor, or neither in $\mathbb{Z}_{50}$? If 20 is a unit, find its inverse. If 20 is zero divisor, show this by finding some $0 \neq m \in \mathbb{Z}_{50}$ such that $20m = 0 \pmod{50}$.

(c) Find all of the principle ideals of $\mathbb{Z}_{50}$.

(d) How many elements generate $\mathbb{Z}_{50}$?
[That is: How many $x \in \mathbb{Z}_{50}$ are there such that $(x) = \langle x \rangle = \mathbb{Z}_{50}$?]

4. (____/15 points) The set $I = (4) = \{0, 4, 8, 12, 16\}$ is an ideal of $\mathbb{Z}_{20}$. Let $R = {\mathbb{Z}_{20}}/{I}$

(a) Write down all of the distinct cosets of I in $\mathbb{Z}_{20}$.

(b) Finish filling out the following addition and multiplication tables for R:

+	+ I	+ I	+ I	+ I
+ I	+ I	+ I	+ I	+ I
+ I	+ I	+ I	+ I	+ I
+ I	+ I	+ I	+ I	+ I
+ I	+ I	+ I	+ I	+ I

×	+ I	+ I	+ I	+ I
+ I	+ I	+ I	+ I	+ I
+ I	+ I	+ I	+ I	+ I
+ I	+ I	+ I	+ I	+ I
+ I	+ I	+ I	+ I	+ I

(c) Fill out the following table of information about R.

Property	Is this true about R?	Briefly Explain
R has a "unity"		
R is commutative		
R is an integral domain		
R is a field		

(d) Fill out the following table concerning R. *Hint:* These tables are longer than they need to be.

Zero Divisors	I am a zero divisor because...

Units	My multiplicative inverse is...

5. (____/12 points) Homomorphisms.

(a) Show $\varphi : \mathbb{Z}_3 \to \mathbb{Z}_6$ defined by $\varphi(x) = 2x$ is a **well-defined** homomorphism.

(b) Find the kernel and image of φ. Is φ one-to-one? onto? an isomorphism?

(c) $S = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \,\middle|\, a, b \in \mathbb{R} \right\}$ is a subring of $M_2(\mathbb{R}) = \mathbb{R}^{2\times 2}$. Define $\psi : S \to \mathbb{R}$ by $\psi\left(\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \right) = a$. Show that ψ is a ring homomorphism.

(d) Find the kernel and image of ψ. Is ψ one-to-one? onto? an isomorphism?

6. (____/12 points) Some random proofs.

(a) Let R be a ring with 1. Let u be a unit of R. Show that u cannot be a zero divisor. (This shows that all fields are integral domains.)

(b) Let R be a commutative ring with 1. Suppose $a \in R$ and recall that $(a) = \{ra \,|\, r \in R\}$ is the principle ideal generated by a. Prove that (a) is an ideal of R.

(c) Let G be a group such that $(xy)^{-1} = x^{-1}y^{-1}$ for all $x, y \in G$. Show that G is abelian.

7. (____/12 points) Just a little harmless permuting.

(a) Let $H = \{(1), (12)\}$. Find all of the **left and right** cosets of H in $S_3 = \{(1), (12), (13), (23), (123), (132)\}$. Is H a normal subgroup?

(b) Explain why A_n (the even permutations) is a normal subgroup of S_n. [Hint: You may either briefly explain *why it is a subgroup* **and** *why it is normal*, or you can find a homomorphism with kernel A_n.]

(c) Consider the quotient group S_n / A_n. It this a cyclic group? Why or why not?

8. (____/12 points) D_{10}. $n = 10$ really? why oh why?

(a) Let G be a **non-abelian** group of order 20. What are possible orders of elements of G? What does "non-abelian" rule out and why?

(b) Half of the elements of D_{10} are reflections and half are rotations. In fact, the **rotations** in D_{10} form a **cyclic subgroup**. Without worrying about the cyclic part, **briefly** explain why the rotations do form a subgroup and why the reflections do not.

(c) Use the description of D_{10} given in part (b) to determine the number of elements of each order in D_{10}.

Element order =

Number of elements with that order =

D_4	1	x	x^2	x^3	y	xy	x^2y	x^3y
1	1	x	x^2	x^3	y	xy	x^2y	x^3y
x	x	x^2	x^3	1	xy	x^2y	x^3y	y
x^2	x^2	x^3	1	x	x^2y	x^3y	y	xy
x^3	x^3	1	x	x^2	x^3y	y	xy	x^2y
y	y	x^3y	x^2y	xy	1	x^3	x^2	x
xy	xy	y	x^3y	x^2y	x	1	x^3	x^2
x^2y	x^2y	xy	y	x^3y	x^2	x	1	x^3
x^3y	x^3y	x^2y	xy	y	x^3	x^2	x	1

Q	1	-1	i	$-i$	j	$-j$	k	$-k$
1	1	-1	i	$-i$	j	$-j$	k	$-k$
-1	-1	1	$-i$	i	$-j$	j	$-k$	k
i	i	$-i$	-1	1	k	$-k$	$-j$	j
$-i$	$-i$	i	1	-1	$-k$	k	j	$-j$
j	j	$-j$	$-k$	k	-1	1	i	$-i$
$-j$	$-j$	j	k	$-k$	1	-1	$-i$	i
k	k	$-k$	j	$-j$	$-i$	i	-1	1
$-k$	$-k$	k	$-j$	j	i	$-i$	1	-1