Math 3110 **Test #2** **October 28^{th}, 2009**

ANSWER KEY

1. (16 points) Either prove G is a group or explain why it is not a group.

(a) $G = \{x \in D_5 \,|\, x \text{ is a reflection}\}$ (the operation is composition of symmetries).

Answer: No. G is not a group.

Two easy ways to see that G is not a group are:
(1) G lacks an identity. The identity symmetry is not a reflection (it's a rotation of zero degrees).
(2) The operation lacks closure. A reflection composed with a reflection is a rotation. So given any $r_1, r_2 \in G$ we have $r_1 r_2 \notin G$.

(b) $\mathbb{R}_{>0}$ (positive real numbers) where the operation is multiplication.

Answer: Yes. $\mathbb{R}_{>0}$ is a group under multiplication.

Notice that $\mathbb{R}_{>0}$ is a subset of $\mathbb{R}_{\neq 0}$ (which is a group under multiplication). So to show that $\mathbb{R}_{>0}$ is a group we may just show it is a subgroup of $\mathbb{R}_{\neq 0}$. To do this we need to check closure under multiplication and closure under inversion.

- $x > 0$ and $y > 0 \Rightarrow xy > 0$ so we have closure under multiplication.
- $x > 0 \Rightarrow 1/x > 0$ so we have closure under inversion.

Therefore, $\mathbb{R}_{>0}$ is a subgroup of $\mathbb{R}_{\neq 0}$ (and so it is a group).

2. (16 points)

(a) Show $H = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \,\middle|\, a, b \in \mathbb{R} \text{ and } a, b \neq 0 \right\}$ is a subgroup of $\mathrm{GL}_2(\mathbb{R})$

(Recall $\mathrm{GL}_2(\mathbb{R})$ — the set of 2×2 invertible matrices — is a group under matrix multiplication).

First, we'll set up some notation. Let $A, B \in H$. Thus there exists some non-zero real numbers a, b, c, d such that $A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ and $B = \begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix}$.
Notice that $\det(A) = ab \neq 0$ because $a \neq 0$ and $b \neq 0$. Therefore, for all $A \in H$, we have $A \in \mathrm{GL}_2(\mathbb{R})$. Thus $H \subset \mathrm{GL}_2(\mathbb{R})$.
Obviously H is not empty.
$AB = \begin{bmatrix} ac & 0 \\ 0 & bd \end{bmatrix}$ (which looks right – it's diagonal). Since a, b, c, d are non-zero, we have that ac and bd are non-zero as well. Therefore, $AB \in H$.
$A^{-1} = \dfrac{1}{ab - 0^2} \begin{bmatrix} b & -0 \\ -0 & a \end{bmatrix} = \begin{bmatrix} 1/a & 0 \\ 0 & 1/b \end{bmatrix}$ (which looks right – it's diagonal). Since a, b are non-zero, we have that $1/a$ and $1/b$ are non-zero as well. Therefore, $A^{-1} \in H$.
Therefore, H is a non-empty subset of $\mathrm{GL}_2(\mathbb{R})$ which is closed under matrix multiplication and inversion. Thus H is a subgroup of $\mathrm{GL}_2(\mathbb{R})$.

(b) Quickly (in a few words) why are the even integers a subgroup of the integers?

Even plus even is even and the negative of an even number is even.

3. (20 points) Workin' mod 5.

(a) Fill out the following tables (don't worry about brackets for equivalence classes.)

$(\mathbb{Z}_5,+)$	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

$\mathbb{Z}_5$ Addition Table

$(\mathbb{Z}_5,\times)$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

$\mathbb{Z}_5$ Multiplication Table

(b) Compute $2^{-1}(4+3)-2 \pmod 5$.

Notice that $2(3)=1 \pmod 5$ so $2^{-1}=3$.
Thus $2^{-1}(4+3)-2 = 3(4+3)-2 = 3(7)-2 = 3(2)-2 = 6-2 = 1-2 = -1 = 4 \pmod 5$
Of course, we could wait until the very end to reduce mod 5: $2^{-1}(4+3)-2 = 3(4+3)-2 = 3(7)-2 = 21-2 = 19 = 4 \pmod 5$. The important thing to remember is that addition, subtraction and multiplication of integers haven't changed. When working mod 5, we've just changed the idea of what "=" means. The only operation that requires special care is "division". To compute inverses we can either guess/look up the answer on the multiplication table OR run the Euclidean algorithm.

(c) Find $\langle 3\rangle$ (the subgroup generated by 3) in $U(5)$ (NOT $\mathbb{Z}_5$!!!).

Remember $U(5)$ contains (congruence classes) of integer which are relatively prime with 5. So $U(5) = \{1,2,3,4\}$ (Also, remember that the operation is multiplication mod 5 NOT addition).
Note: $3^1=3$, $3^2=4$, $3^3=3^2(3)=4(3)=2$, and $3^4=3^3(3)=2(3)=1$ so the order of 3 (in $U(5)$) is 4.

$$\langle 3\rangle = \{\dots 3^{-1},3^0,3^1,3^2,\dots\} = \{1,3,4,2\} = U(5)$$

(d) Find the orders of elements of $U(5)$. Is $U(5)$ cyclic? Why or why not?

- $1^1 = 1$
- $2^1=2$, $2^2=4$, $2^3=3$, $2^4=1$
- $3^1=3$, $3^2=4$, $3^3=2$, $3^4=1$
- $4^1=4$, $4^2=1$

element =	1	2	3	4
order =	1	4	4	2

YES $U(5)$ is cyclic because the order of $U(5)$ is 4 and we have an element (in fact 2 elements) of order 4. ...OR... $U(5)$ is cyclic because $U(5)=\langle 3\rangle$ ...OR... $U(5)$ is cyclic because $U(5)=\langle 2\rangle$.

4. (16 points) Quick proofs

(a) Let G be a group and suppose that $x=x^{-1}$ for all $x\in G$. Prove that G is Abelian.
Hint: $xy=(xy)^{-1}=$??

Suppose that $x,y\in G$. Then $x^{-1}=x$, $y^{-1}=y$ and $(xy)=(xy)^{-1}$ (by assumption). Recall that in any group we have $(xy)^{-1}=y^{-1}x^{-1}$ (the "socks shoes principle"). Therefore, $xy=(xy)^{-1}=y^{-1}x^{-1}=yx$. Thus G is Abelian.

(b) Let n be an integer such that $n\geq 3$. Consider $(12),(13)\in S_n$. Compute $(12)(13)$ and $(13)(12)$. Is S_n cyclic? Why or why not?

- $(12)(13) = (132)$
- $(13)(12) = (123)$

Since $(12)(13)\neq(13)(12)$, we can conclude that S_n is **not Abelian**. And since S_n is not Abelian, it is **not cyclic** (because cyclic implies Abelian).

5. (16 points) $G = \{1, a, a^2, b, ab, a^2b\}$ is a group.
Finish filling out G's Cayley table then answer some questions.

G	1	a	a^2	b	ab	a^2b
1	1	a	a^2	b	ab	a^2b
a	a	a^2	1	ab	a^2b	b
a^2	a^2	1	a	a^2b	b	ab
b	b	a^2b	ab	1	a^2	a
ab	ab	b	a^2b	a	1	a^2
a^2b	a^2b	ab	b	a^2	a	1

(a) What is the order of a^2? Determine $\langle a^2 \rangle$ (the subgroup generated by a^2).

$(a^2)^1 = a^2$, $(a^2)^2 = a$, and $(a^2)^3 = a(a^2) = 1$. Thus the order of a^2 is 3.
Also, $\langle a^2 \rangle = \{a^2, a, 1\} = \{1, a, a^2\}$.

(b) Is G Abelian? Is G cyclic? Why or why not?

NO G is **not Abelian**. Just look at the Cayley table and notice that $ab \neq ba = a^2b$. Also, since G is not Abelian it is **not cyclic** (since all cyclic groups are automatically Abelian).

6. (16 points) Permutations!

(a) Write $\sigma = (125)(35)(24)(264)$ as a product of disjoint cycles.

$\sigma = (125)(35)(24)(264) = (12653)(4) = (12563)$

(b) What is the order of σ?

The order of σ is 5 (the least common multiple of the lengths of the cycles when σ is written as a product of **disjoint** cycles).

(c) Write σ as a product of transpositions. Is σ even or odd?

There are infinitely many answers to this question here are two possibilities:
$\sigma = (15)(12)(35)(24)(24)(26) = (13)(16)(15)(12)$.

Either way we have an even number of transpositions so σ **is even**.

(d) Let $\tau = (1452)(367)(89)$. What is the order of τ?

τ is written as the product of disjoint cycle so we can just take the least common multiple of the lengths of its cycles. $\text{lcm}(4, 3, 2) = 12$. So the order of τ is 12.

(e) Write τ^{26} as the product of disjoint cycles.

Since the order of τ is 12, we have that $\tau^{12} = (1)$ so $\tau^{26} = \tau^{12}\tau^{12}\tau^2 = \tau^2 = (15)(24)(376)$.

(f) What is the order of τ^6?
[Hint: You shouldn't need to compute any powers of τ to answer this question.]

Quick way: $\tau^6 \neq (1)$ since the order of τ is greater than 6. But $(\tau^6)^2 = \tau^{12} = (1)$ since the order of τ is 12. Thus the second power of τ^6 is the identity so τ^6 has order 2.

Calculatin' way: $\tau^6 = \left((1452)(367)(89)\right)^6 = (1452)^6(367)^6(89)^6$ since disjoint cycles commute. Continuing... $= (1456)^4(1456)^2((367)^3)^2((89)^2)^3 = (1)(15)(24)(1)^2(1)^3 = (15)(24)$ so τ^6 has order $\text{lcm}(2, 2) = 2$.