Math 3110 **Test #3** **November 20^{th}, 2009**

ANSWER KEY

1. (16 points) The following groups are **NOT** isomorphic. Explain why.

(a) $D_5 \not\cong \mathbb{Z}_{10}$

Notice that both of these groups are of order 10, but that's about the only property that matches.

Here are a few possible answers:

- D_5 is not abelian but $\mathbb{Z}_{10}$ is abelian, so they cannot be isomorphic.
- D_5 is not abelian thus not cyclic but $\mathbb{Z}_{10}$ is cyclic, so they cannot be isomorphic.
- D_5 has elements of order 1, 2, and 5. On the other hand, $\mathbb{Z}_{10}$ has elements of order 1, 2, 5, and 10. Thus they cannot be isomorphic.
- D_5 has 5 elements of order 2 but $\mathbb{Z}_{10}$ only has 1 element of order 2, so they cannot be isomorphic.

(b) $D_4 \not\cong Q$ where $Q = \{\pm 1, \pm i, \pm j, \pm k\}$ (the quaternions).

Both D_4 and Q are non-abelian groups of order 8. So we need to look elsewhere. Let's try looking at the order of each element.

D_4				Q			
order =	1	2	4	order =	1	2	4
elements =	1	x^2, y, xy, x^2y, x^3y	x, x^3	element =	1	-1	$\pm i, \pm j, \pm k$
# of elements =	1	5	2	# of elements =	1	1	6

This shows us that these groups cannot be isomorphic. For example, D_4 has 5 elements of order 2 while Q only has 1 element of order 2.

2. (14 points) Is it possible?

(a) Let G_1 and G_2 be groups of order 7. Is it possible that $G_1 \not\cong G_2$? If so, give examples of non-isomorphic groups of order 7. If G_1 is always isomorphic to G_2, explain why.

7 is prime. We know that groups of prime order are cyclic and any two cyclic groups of the same order are isomorphic. Therefore, G_1 and G_2 **must** be isomorphic. So "No" $G_1 \not\cong G_2$ is not possible.

(b) Let G_1 and G_2 be groups of order 6. Is it possible that $G_1 \not\cong G_2$? If so, give examples of non-isomorphic groups of order 6. If G_1 is always isomorphic to G_2, explain why.

"Yes" it is possible that $G_1 \not\cong G_2$. For example, let $G_1 = \mathbb{Z}_6$ and $G_2 = D_3$. These groups are not isomorphic because G_1 is abelian and G_2 is not.

3. (18 points) Homomorphisms

(a) Quickly, explain why $\psi : \mathbb{Z}_3 \to \mathbb{Z}_7$ defined by $\psi(x) = x$ is **NOT** a well-defined function.

"Well-defined" means "equal inputs implies equal outputs".
Notice that $0 = 3 \pmod 3$ but $\psi(0) = 0 \neq 3 = \psi(3) \pmod 7$. So ψ is not well-defined.

(b) Let $\varphi : \mathbb{Z}_8 \to \mathbb{Z}_{12}$ be defined by $\varphi(x) = 6x$. Show that φ is a well-defined homomorphism (i.e. show that φ is well-defined *and* φ is a homomorphism).

Well-defined Suppose $x = y \pmod 8$. Then $x = y + 8k$ for some $k \in \mathbb{Z}$. $\varphi(x) = 6x = 6(y + 8k) = 6y + 48k = 6y + 12(4k)$ so $\varphi(x) = 6x = 6y = \varphi(y) \pmod{12}$.

Homomorphism $\varphi(x + y) = 6(x + y) = 6x + 6y = \varphi(x) + \varphi(y)$ for all $x, y \in \mathbb{Z}_8$

(c) Using the homomorphism φ from part (b), find $\text{Ker}(\varphi)$ and $\varphi(\mathbb{Z}_8)$.

$\varphi(0) = 0$, $\varphi(1) = 6$, $\varphi(2) = 0$, $\varphi(3) = 6$, $\varphi(4) = 0$, $\varphi(5) = 6$, $\varphi(6) = 0$, and $\varphi(7) = 6$.

So we have that $\text{Ker}(\varphi) = \{0, 2, 4, 6\}$ and $\varphi(\mathbb{Z}_8) = \{0, 6\}$. Remember that the size of the kernel times the size of the image should be the size of the domain — this holds here $4 \times 2 = 8$.

Is φ one-to-one? No. The kernel is non-trivial. This is a 4-to-1 map.

Is φ onto? No. The image is not equal to the codomain: $\{0, 6\} \neq \mathbb{Z}_{12}$

Is φ an isomorphism? No. To be an isomorphism the homomorphism would need to be 1-1 and onto. It's neither!

4. (16 points) Write down the left multiplication operators for $U(8) = \{1, 5, 3, 7\}$ then translate them into permuations (in S_4) labeling the elements of $U(8)$ as follows:

$$1 \rightsquigarrow 1, \qquad 5 \rightsquigarrow 2, \qquad 3 \rightsquigarrow 3, \qquad \text{and} \qquad 7 \rightsquigarrow 4.$$

Finally, write down a subgroup of S_4 which isomorphic to $U(8)$ (using Cayley's theorem).

Let $g, x \in U(8)$. We write $L_g(x) = gx$ (so that L_g is the left multiplication by g operator). We have (for each $g \in U(8)$) L_g's action on $U(8)$ and then this action translated to the proper labels $1, \ldots, 4$:

	in $U(8)$			in S_4		
$L_1:$	1	$\mapsto$	1	1	$\mapsto$	1
	5	$\mapsto$	5	2	$\mapsto$	2
	3	$\mapsto$	3	3	$\mapsto$	3
	7	$\mapsto$	7	4	$\mapsto$	4
$L_5:$	1	$\mapsto$	5	1	$\mapsto$	2
	5	$\mapsto$	1	2	$\mapsto$	1
	3	$\mapsto$	7	3	$\mapsto$	4
	7	$\mapsto$	3	4	$\mapsto$	3

	in $U(8)$			in S_4		
$L_3:$	1	$\mapsto$	3	1	$\mapsto$	3
	5	$\mapsto$	7	2	$\mapsto$	4
	3	$\mapsto$	1	3	$\mapsto$	1
	7	$\mapsto$	5	4	$\mapsto$	2
$L_7:$	1	$\mapsto$	7	1	$\mapsto$	4
	5	$\mapsto$	3	2	$\mapsto$	3
	3	$\mapsto$	5	3	$\mapsto$	2
	7	$\mapsto$	1	4	$\mapsto$	1

Therefore we have the following map $\psi : U(8) \to S_4$:

$1 \mapsto (1) \qquad 5 \mapsto (12)(34) \qquad 3 \mapsto (13)(24) \qquad 7 \mapsto (14)(23)$

By Cayley's theorem: $U(8) \cong \{(1), (12)(34), (13)(24), (14)(23)\}$.

5. (18 points) Quotient Groups

(a) Let $H = \{0, 4, 8\} \lhd \mathbb{Z}_{12}$. Write out all of the cosets of H.
Then fill out a Cayley table for the quotient $\mathbb{Z}_{12}/H$.

Remember the operation in $\mathbb{Z}_{12}$ is addition modulo 12, so we have additive cosets. Also, $\mathbb{Z}_{12}$ is abelian, so left and right cosets are equal.

$H = \{0, 4, 8\} = 0 + H = 4 + H = 8 + H$, $1 + H = \{1, 5, 9\} = 5 + H = 9 + H$, $2 + H = \{2, 6, 10\} = 6 + H = 10 + H$, and $3 + H = \{3, 7, 11\} = 7 + H = 11 + H$.

$\mathbb{Z}_{12}/H$	H	$1+H$	$2+H$	$3+H$
H	H	$1+H$	$2+H$	$3+H$
$1+H$	$1+H$	$2+H$	$3+H$	H
$2+H$	$2+H$	$3+H$	H	$1+H$
$3+H$	$3+H$	H	$1+H$	$2+H$

Sample Calculations: $H+(3+H) = (0+H)+(3+H) = (0+3)+H = 3+H$ and $(2+H)+(3+H) = (2+3)+H = 5+H = \{1,5,9\} = 1+H$

(b) Consider $K = \{1, x^2\}$ in D_4. Then $xK \in D_4/K$.
Compute $(xK)^2 = xK\,xK$. What is the order of xK (as an element of this quotient group)?

$(xK)^2 = xKxK = x^2K = \{x^2 \cdot 1, x^2 \cdot x^2\} = \{x^2, 1\} = K$

Since xK is not the identity of the quotient group ($xK = \{x, x^3\} \neq K$) but $(xK)^2 = K$, we have that xK is an element of order 2. [Note: As a set, xK has size 2 as well – this is just a coincidence.]

6. (18 points) Other stuff.

(a) Let H be a subgroup of G and K be a subgroup of H ($K \underset{\text{s.g.}}{\subseteq} H \underset{\text{s.g.}}{\subseteq} G$).

If $|G| = 18$, then what are the possible orders of H?

By Lagrange's theorem, the order of H must divide the order of G. Thus the possible orders of H are: $1, 2, 3, 6, 9$, and 18.

Suppose that $H \neq G$ and H has more than 7 elements. Also, K contains more than just the identity. What are the possible orders of K?
[Hint: determine the order of H first, then remember that K is a subgroup of H.]

$H \neq G$ says that $|H| < 18$. But we are also told that $|H| > 7$. The only possibility left on our list is $|H| = 9$. Now K is a subgroup of H. Therefore, by Lagrange's theorem the order of K must divide the order of H (which is 9). Thus $|K| = 1, 3$, or 9. Finally, we are told that K contains more than just the identity, so $|K| > 1$.

Answer: The order of K is either 3 or 9 (In fact, if $|K| = 9$, then $K = H$).

(b) Let G be an **Abelian** group and $H \underset{\text{s.g.}}{\subseteq} G$. Give a quick explanation why H is **normal** in G and then prove that G/H is Abelian.

Let $g \in G$. Then $gH = \{gh \,|\, h \in H\} = \{hg \,|\, h \in H\} = Hg$ since $gh = hg$ for all $g, h \in G$ since G is abelian. Thus H is normal in G. Alternatively, just notice that $ghg^{-1} = hgg^{-1} = h$ since G is abelian. Thus H is closed under conjugation and hence is normal. Now that we know H is a normal subgroup, it's ok to talk about the quotient of G by H.

Let $aH, bH \in G/H$. $aHbH = abH = baH = bHaH$ since $ab = ba$ for all $a, b \in G$ because G is abelian. Therefore, G/H is abelian.