**Math 3110** | # Test #2 | **October $26^{\text{th}}$, 2011**

**Name:** ______________________________ **Be sure to show your work!**

1. (____/20 points) Cyclic

(a) Let $G = \langle g \rangle$ where $g$ has order 20.

$\langle g^8 \rangle =$ ______________________________

Is $g^{102} \in \langle g^8 \rangle$? Why or why not?

(b) Suppose $G$ is a cyclic group with at least one element of order 6.

What can you say about the order of $G$?

How many elements of order 6 can $G$ have? Is there more than one possibility?

(c) List the possible orders of elements in $\mathbb{Z}_{33}$. Then determine the number of elements of each order.

| | | | | | | |
|---|---|---|---|---|---|---|
| Order = | | | | | | |
| Number of elements = | | | | | | |

Now fill out a table for $D_{33}$.

| | | | | | | |
|---|---|---|---|---|---|---|
| Order = | | | | | | |
| Number of elements = | | | | | | |

2. (____/20 points) The following pairs of groups are **not** isomorphic. Prove this is the case.

(a) $\mathrm{GL}_3(\mathbb{R}) \ncong A_{500}$

(b) $U(8) = \{1, 3, 5, 7\} \ncong \mathbb{Z}_4$

(c) $\mathrm{GL}_2(\mathbb{Z}) \ncong \mathbb{Q}$

(d) $S_4 \ncong D_{12}$

3. (____/20 points) Isomorphisms

(a) Prove that $U(5) \cong \mathbb{Z}_4$.

(b) Let $G$ be an **Abelian** group. Define the map $\varphi : G \to G$ by $\varphi(g) = g^{-1}$. Prove that $\varphi$ is an isomorphism (actually $\varphi$ is an automorphism since its domain and codomain are equal).

Is $\varphi$ an automorphism if $G$ is not Abelian? Why or why not?

4. (____/20 points) Zombie Apocalypse! Does anyone actually read the directions to these problems?

(a) Let $\sigma = (2453)(1346)(126) \in S_6$

Write $\sigma$ as a product of disjoint cycles

Find $\sigma^{-1}$

What is the order of $\sigma$? ______________________________

Is $\sigma$ even or odd? ______________________________

(b) For convenience: $A_4 = \{(1), (123), (132), (124), (142), (134), (143), (234), (243), (12)(34), (13)(24), (14)(23)\}$

Consider the subgroup $H = \{(1), (12)(34), (13)(24), (14)(23)\}$. Find the left cosets of $H$ in $A_4$.

(c) Let $\sigma = (14)(23) \in S_4$. What is the order of $\sigma$? ______________________________

Compute $\sigma^{999}$.

(d) Find an element of order 15 in $S_8$.

5. (____/20 points)

(a) Let $G$ be a group, $H$ be a subgroup of $G$, and $a, b \in G$. Prove $aH = bH$ implies that $a^{-1}b \in H$.
[You may **not** use any theorems about cosets.]

(b) My friend is computing some cosets of $K$ which is a subgroup of $S_4$. He claims has found a left coset $L = \{(243), (142), (123), (134)\}$.

Assuming my friend didn't make a mistake, what is the order of $K$? ______________________________

How many cosets will $K$ have in $S_4$? ______________________________

My friend then starts computing right cosets and finds a coset $R = \{(1234), (24), (1432), (13), (1324)\}$.
I know he must have made a mistake. Why?

(c) $H$ and $K$ are subgroups of $G$ such that $H \subsetneq K \subsetneq G$ (they are proper subsets of each other).
$G$ has order 50. $K$ has order 5. What are the possible orders for $H$?

*Note:* This is the test as it was given. However, the intended problem is: $G$ has order 50. $H$ has order 5. What are the possible orders for $K$? [switch $H$ and $K$.]