Math 3110 # Test #3 **November 18^{th}, 2011**

Name: Answer Key **Be sure to show your work!**

1. (25 points) 3-2-1...Go!

(a) How many elements of order 3 are in $\mathbb{Z}_{9000} \oplus \mathbb{Z}_{3333333}$?

The order of an element of $\mathbb{Z}_{9000} \oplus \mathbb{Z}_{3333333}$ is computed by finding the least common multiple of the orders of each of its components: $|(a,b)| = \text{lcm}(|a|,|b|)$. If we have to have an element of order 3, then $3 = |(a,b)| = \text{lcm}(|a|,|b|)$. The only ways to get an lcm of 3 are $\text{lcm}(3,1) = \text{lcm}(1,3) = \text{lcm}(3,3) = 3$.

Now both $\mathbb{Z}_{9000}$ and $\mathbb{Z}_{3333333}$ are cyclic groups of orders divisible by 3. So they both have exactly 1 element of order 1 (the identity) and 2 elements of order 3 (the two generators of the unique subgroup of order 3). Thus there are $2 \cdot 1 + 1 \cdot 2 + 2 \cdot 2 = 8$ elements of order 3.

Alternatively, taking all elements of order 1 or 3, we have $3 \cdot 3 = 9$ choices. Only the identity has order 1, so that leaves exactly $9 - 1 = 8$ elements of order 3.

Unnecessary Extra: By the way, if 3 divides n, then $n/3$ and $n/3 + n/3 = 2n/3$ are the elements of order 3 in $\mathbb{Z}_n$. So since 0 has order 1 in both groups, 3000 and 6000 have order 3 in $\mathbb{Z}_{9000}$, and 1111111 and 2222222 have order 3 in $\mathbb{Z}_{3333333}$. Thus the elements of order 3 are: $(3000,0)$, $(6000,0)$, $(0,1111111)$, $(0,2222222)$, $(3000,1111111)$, $(3000,2222222)$, $(6000,1111111)$, and $(6000,2222222)$.

(b) Let H be a subgroup of G where G is abelian. Quickly explain why H is a normal subgroup of G.

Let $a \in G$. Then $aH = \{ah \mid h \in H\} = \{ha \mid h \in H\} = Ha$ (the middle equality holds because G is abelian).

Alternate proof: Let $a \in G$ and $h \in H$. Then $aha^{-1} = haa^{-1} = h \in H$ (the first equality again follows from the fact that G is abelian). Thus $aHa^{-1} \subseteq H$ so H is normal.

(c) Consider for example: $\mathbb{Z}_{12}/\langle 4 \rangle \cong \mathbb{Z}_4$. Now let $k\ell = n$ (where k, ℓ, n are positive integers) and let $G = \mathbb{Z}_n/\langle k \rangle$. Briefly, explain why the G has order k, why G is cyclic, and why $G \cong \mathbb{Z}_k$.

Recall that $|\langle k \rangle| = n/k$ if k divides n. Therefore, $|G| = |\mathbb{Z}_n|/|\langle k \rangle| = \dfrac{n}{n/k} = \dfrac{n}{\ell} = k$. G is cyclic because G is a quotient of $\mathbb{Z}_n$ and any quotient of a cyclic group is cyclic. Finally, G and $\mathbb{Z}_k$ are both cyclic groups of order k. Thus they must be isomorphic (any two cyclic groups of the same order are isomorphic).

(d) Let G and H be group. Prove that $\pi : G \oplus H \to G$ defined by $\pi((g,h)) = g$ is a homomorphism which is onto. Find the kernel of π. Finally, show that $G \oplus H / \{e\} \oplus H \cong G$.

Let $(g,h),(x,y) \in G \oplus H$. $\pi((g,h)(x,y)) = \pi((gx,hy)) = gx = \pi((g,h))\pi((x,y))$. Thus π is a homomorphism. Next, let $g \in G$. Since H is a group, it contains at least an identity element, say $e \in H$. Then $(g,e) \in G \oplus H$ and $\pi((g,e)) = g$. Therefore, π is onto. $\ker(\pi) = \{(g,h) \in G \oplus H \mid \pi((g,h)) = e\} = \{(g,h) \in G \oplus H \mid g = e\} = \{(e,h) \mid h \in H\} = \{e\} \oplus H$. Finally, noting that $\ker(\pi) = \{e\} \oplus H$ and $\text{image}(\pi) = G$ (since π is onto), we have (by the first isomorphism theorem) that $G \oplus H / \{e\} \oplus H = G \oplus H / \ker(\pi) \cong \text{image}(\pi) = G$.

2. (25 points) Quotients

(a) Given: $K = \{R_0, R_{180}\}$ is a normal subgroup of D_6.

The order of D_6/K is $|D_6|/|K| = 12/2 = 6$.

The identity of D_6/K is K (or equivalently $R_0K = R_{180}K = \{R_0, R_{180}\}$).

$(R_{60}K)^{-1} = R_{60}^{-1}K = R_{-60}K = R_{300}K = R_{120}K = \{R_{120}, R_{300}\}$.

The order of $R_{60}K$ in ${D_6}/{K}$ is $\underline{\ \ 3\ \ }$. The size of the set $R_{60}K$ is $\underline{\ \ 2\ \ }$.

Scratch work:

$R_{60}K$ contains two elements because all left cosets have the same size (i.e. that of the subgroup K).

$R_{60}K \neq K$, $(R_{60}K)^2 = R_{60}^2K = R_{120}K \neq K$, $(R_{60}K)^3 = R_{60}^3K = R_{180}K = K$. Thus the order of $R_{60}K$ is 3.

(b) Let H be a (normal) subgroup of G where G is abelian. Prove that ${G}/{H}$ is abelian.

Let $aH, bH \in {G}/{H}$. Then $aHbH = abH = baH = bHaH$ where $ab = ba$ because G is abelian.

(c) Consider ${\mathbb{Z}_{12}}/{H}$ where $H = \langle 4 \rangle = \{0, 4, 8\}$. List all of the cosets (and their contents). Then make a Cayley table for this quotient group.

There are $|\mathbb{Z}_{12}|/|H| = 12/3 = 4$ cosets.

- $H = \{0, 4, 8\}$
- $1 + H = \{1, 5, 9\}$
- $2 + H = \{2, 6, 10\}$
- $3 + H = \{3, 7, 11\}$

Example: $(2 + H) + (3 + H) = (2 + 3) + H = 5 + H = 1 + H$ (since 1 and 5 belong to the same coset).

	H	$1+H$	$2+H$	$3+H$
H	H	$1+H$	$2+H$	$3+H$
$1+H$	$1+H$	$2+H$	$3+H$	H
$2+H$	$2+H$	$3+H$	H	$1+H$
$3+H$	$3+H$	H	$1+H$	$2+H$

Bonus Problem: I almost put this on the test and then got rid of it because of length.

(d) Let G be a group. A group H is called a *homomorphic image* of G if there exists a homomorphism $\varphi : G \to H$ which is onto (i.e. the image of φ is H). Explain why homomorphic images and quotients of G share the same group properties. Specifically why is it that if G has an abelian quotient of order 123 if and only if G has an abelian homomorphic image of order 123).

Suppose that H is a homomorphic image of G. Then there is a homomorphism $\varphi : G \to H$ which is onto. Therefore, by the first isomorphism theorem, ${G}/{\ker(\varphi)} \cong H$. Thus H is (isomorphic to) a quotient of G. Conversely, any quotient of G is the image of the corresponding projection homomorphism: $\pi : G \to {G}/{K}$ defined by $\pi(g) = gK$ (π is an onto homomorphism and its kernel is K).

Therefore if, for example, G has an abelian quotient of order 123, say G/K is abelian of order 123. Then $\pi : G \to G/K$ is a homomorphism from G onto an abelian group of order 123. Thus G has a homomorphic image which is abelian of order 123.

3. (25 points) Oh no! I've made a mistake. $2^2 \cdot 5 \neq 50$

In each of the following situations, explain <u>why</u> we know **a mistake has been made**.
(Why are these statements wrong?)

(a) We found a homomorphism $\varphi : \mathbb{Z}_{50} \oplus \mathbb{Z}_{10} \to D_5$ which is onto.

If there was an onto homomorphism from $\mathbb{Z}_{50} \oplus \mathbb{Z}_{10}$ to D_5. Then (by the first isomorphism theorem) we would have that there is a quotient of $\mathbb{Z}_{50} \oplus \mathbb{Z}_{10}$ which is isomorphic to D_5. But this is impossible, quotients of abelian groups are abelian and D_5 is not abelian.

OR, briefly, an abelian group cannot have a non-abelian homomorphic image.

(b) Let $H = Z(D_6) = \{R_0, R_{180}\}$ (the center of D_6). After some shoddy computations, I've found that $D_6/H \cong \mathbb{Z}_6$. [*Hint:* No computations needed to shoot this down.]

The G/Z theorem states that if a group quotiented by its center is cyclic, then the group must be abelian. $\mathbb{Z}_6$ is cyclic. But D_6 is not abelian, so this is impossible.

(c) I just found a normal subgroup $H \triangleleft S_4$ such that S_4/H is an abelian group of order 18.

$|S_4| = 4! = 24$. We are told that $|S_4/H| = |S_4|/|H| = 24/|H| = 18$. There is no way to divide 24 by a postive integer and get 18. So this is impossible. OR, briefly, there is no quotient of order 18 because 18 is not a divisor of $|S_4| = 4! = 24$.

(d) My friend Herbert found an element $(x, y) \in D_6 \oplus \mathbb{Z}_{10}$ whose order is 60.

In a direct product of groups we have $|(x, y)| = \text{lcm}(|x|, |y|)$. $x \in D_6$ so its order is $1, 2, 3$, or 6. $y \in \mathbb{Z}_{10}$ so its order is $1, 2, 5$, or 10. The largest possible order (coming from these choices) is $\text{lcm}(6, 10) = 30$. There are no elements of order 60 in this group.

4. (25 points) Finite Abelian Groups

(a) List all of the non-isomorphic abelian groups of order $100 = 2^2 5^2$. (*Note*: $4 \cdot 25 = 100 \leftarrow$ I can multiply!)

There are 2 partitions of 2: $2 = 2$ and $2 = 1 + 1$. Thus there are 2 abelian groups of order $2^2 = 4$ and 2 abelian groups of order $5^2 = 25$. Therefore, there are a total of $2 \cdot 2 = 4$ non-isomorphic abelian groups of order 100.

- $\mathbb{Z}_4 \oplus \mathbb{Z}_{25}$ ($\cong \mathbb{Z}_{100}$)
- $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{25}$ ($\cong \mathbb{Z}_2 \oplus \mathbb{Z}_{50}$)
- $\mathbb{Z}_4 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_5$ ($\cong \mathbb{Z}_5 \oplus \mathbb{Z}_{20}$)
- $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_5$ ($\cong \mathbb{Z}_{10} \oplus \mathbb{Z}_{10}$)

(b) How many non-isomorphic abelian groups of order 449,878,000 are there?

Note: $449{,}878{,}000 = 2^4 5^3 11^3 13^2$ and there are 5 non-isomorphic abelian groups of order $2^4 = 16$. ☺

There are 5 abelian groups of order 2^4, 3 abelian groups of order 5^3, 3 abelian groups of order 11^3, and 2 abelian groups of order 13^2. Multiplying these (independent choices) together we get $5 \cdot 3 \cdot 3 \cdot 2 = 90$ non-isomorphic abelian groups of order $2^4 5^3 11^3 13^2$ [I think I'll skip listing these ☺].

(c) Are the groups $\mathbb{Z}_{18} \oplus \mathbb{Z}_{12} \oplus \mathbb{Z}_{25}$ and $\mathbb{Z}_{180} \oplus \mathbb{Z}_{30}$ isomorphic? Explain your answer.

Note that both groups are of the same order: $18 \cdot 12 \cdot 25 = 5400 = 180 \cdot 30$. There are many ways to go about answering this question. I will break up both groups into cyclic groups of prime power orders and then compare.

For the first group, $18 = 2 \cdot 3^2$, $12 = 2^2 \cdot 3$, and $25 = 5^2$. Therefore, $\mathbb{Z}_{18} \oplus \mathbb{Z}_{12} \oplus \mathbb{Z}_{25} \cong \mathbb{Z}_9 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_{25}$

For the second group, $180 = 2^2 \cdot 3^2 \cdot 5$ and $30 = 2 \cdot 3 \cdot 5$. Therefore, $\mathbb{Z}_{180} \oplus \mathbb{Z}_{30} \cong \mathbb{Z}_4 \oplus \mathbb{Z}_9 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5$

Both groups have $\mathbb{Z}_2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_9$ in common. However, $\mathbb{Z}_{25}$ cannot be split into $\mathbb{Z}_5 \oplus \mathbb{Z}_5$. Therefore, these groups are not isomorphic [for example, the first group has elements of order 25 and the second does not.]

(d) Is the group $\mathbb{Z}_{15} \oplus \mathbb{Z}_7 \oplus \mathbb{Z}_{121}$ cyclic? (Note: $121 = 11^2$) Explain you answer.

Yes. Notice that $15 = 3 \cdot 5$, 7, and $121 = 11^2$ are relatively prime (they don't share any prime factors). Therefore, $\mathbb{Z}_{15} \oplus \mathbb{Z}_7 \oplus \mathbb{Z}_{121} \cong \mathbb{Z}_{3 \cdot 5 \cdot 7 \cdot 11^2} = \mathbb{Z}_{12705}$ (which is cyclic).