Math 3110 # Big Quiz #1 **September 23^{rd}, 2013**

Name: ________________________________ **Be sure to show your work!**

1. (20 points) Definition and Basics

(a) Suppose that G is a non-empty set equipped an operation. What 4 things do I need to check to see if G is a group?

1:

2:

3:

4:

What additional property needs to hold for G to be an **Abelian** group?

5:

(b) Let $G = \mathbb{Q}_{\leq 0}$ be the non-positive rational numbers. Prove G is **not** a group under addition.

(c) Let $G = \mathbb{R}$ (the real numbers). Prove G is **not** a group under subtraction.

2. (20 points) Working mod 20.

(a) What is the inverse of 3 in the group $\mathbb{Z}_{20}$? What operation makes $\mathbb{Z}_{20}$ a group?

(b) Is 3 an element of $U(20)$? If not, why not? If so, why so & what is its inverse?

(c) List all of the *distinct* cyclic subgroups, $\langle x \rangle$, of $\mathbb{Z}_{20}$.

(d) What is the order of 15 in $\mathbb{Z}_{20}$?

(e) Draw the subgroup lattice of $\mathbb{Z}_{20}$ [Note: $20 = 2^2 \cdot 5$].

3. (20 points) More Modular Arithmetic.

(a) List the elements of $U(8)$. Then find their **order**s and the list the elements in **cyclic subgroup** generated by that element. [*Note:* There may be more spaces than you need.]

$x =$						
$\lvert x \rvert =$						
$\langle x \rangle =$	$\{\quad\}$	$\{\quad\}$	$\{\quad\}$	$\{\quad\}$	$\{\quad\}$	$\{\quad\}$

(b) Is $U(8)$ cyclic? **Yes** / **No** (Circle the correct answer.)

(c) Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. Is $A \in \mathrm{GL}_2(\mathbb{Z}_7)$? If so, find A^{-1}. If not, explain why not.

(d) Find 36^{-1} mod 151 using the extended Euclidean algorithm.

4. (20 points) Recall that $D_4 = \{1, x, x^2, x^3, y, xy, x^2y, x^3y\} = \langle x, y \mid x^4 = 1, y^2 = 1, (xy)^2 = 1\rangle$.

(a) Use the relations for D_4 to derive the relation: $yx = x^{-1}y$.

(b) Fill in the Cayley table for D_4:

	1	x	x^2	x^3	y	xy	x^2y	x^3y
1	1	x	x^2	x^3	y	xy	x^2y	x^3y
x	x							
x^2	x^2							
x^3	x^3							
y	y							
xy	xy							
x^2y	x^2y							
x^3y	x^3y							

(c) Find the inverse and order of each element in D_4.

$g =$	1	x	x^2	x^3	y	xy	x^2y	x^3y
$g^{-1} =$								
$\|g\| =$								

(d) Do the rotations form a subgroup of D_4? If so, why? If not, why not?

5. (20 points) Proofs!

(a) Choose one of the following:

I. Prove that $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(x) = 5x$ is 1-1 but not onto.

II. Let G be a group and $g \in G$. Prove that $|g| = |g^{-1}|$ (i.e. g and its inverse have the same order).

(b) Choose one of the following: (You **must** use a subgroup test in your proof.)

I. Prove that $\mathrm{SL}_2(\mathbb{Z}) = \left\{A \in \mathbb{Z}^{2\times 2} \mid \det(A) = 1\right\}$ is a subgroup of $\mathrm{GL}_2(\mathbb{Z})$.

II. Prove that $H = \{n \in \mathbb{Z} \mid n = 10x + 6y \text{ for some } x, y \in \mathbb{Z}\}$ is a subgroup of $\mathbb{Z}$.