Math 3110 # Midterm Exam — In Class **October 14^{th}, 2013**

Name: ________________________________ **Be sure to show your work!**

1. (12 points) Basics

(a) Let $S = (0, 1] = \{x \mid 0 < x \leq 1\}$. Then S is **not** a group under multiplication. List the group axioms which hold and then using **concrete** counterexamples, show the other axioms fail.

Axioms that hold:

Axioms that fail:

(b) Let $\mathbb{E}$ be the set of even integers. Show $\mathbb{E}$ is a subgroup of $\mathbb{Z}$ using a subgroup test.

2. (14 points) Cyclic Stuff

(a) Let G be a finite group and $g \in G$. Suppose that $|g| = 30$.

i. What is the order of g^{25}? List the distinct elements in $\langle g^{25} \rangle$.

$$\langle g^{25} \rangle = \Bigg\{ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \Bigg\}$$

ii. Is $g^{904} \in \langle g^{25} \rangle$? **Yes** / **No**

(b) How many elements of order 4 does $\mathbb{Z}_{100}$ have? What are they?

(c) Draw the subgroup lattice for $\mathbb{Z}_{75}$. [*Note:* $75 = 3 \cdot 5^2$.]

3. (13 points) Permutations

(a) What is the order of $\sigma = (142)(235)(13)(25)$?

(b) Let $\sigma = (134)(24)(12)$. Find σ^{-1}.

(c) Write $\sigma = (1572)(2345)(12)$ as a product of transpositions. σ is **Even** / **Odd**

(d) Let $\sigma = (12345)(67)$. Compute σ^{995}.

(e) Does S_9 have an element of order 14? If so, give an example. If not, explain why not.

4. (12 points) Explain why the following pairs of groups are not isomorphic.

(a) $\mathrm{GL}_2(\mathbb{R}) \ncong \mathbb{C}$

(b) $U(8) \ncong \mathbb{Z}_4$

(c) $A_4 \ncong D_6$

5. (12 points) Prove that the following pairs of groups are isomorphic.

(a) $U(5) \cong \mathbb{Z}_4$

(b) Consider $\mathbb{R}_{>0}$ (positive reals under multiplication) and $\mathbb{R}$ (under addition). Show $\mathbb{R}_{>0} \cong \mathbb{R}$.
[*Hint:* Consider $\varphi(x) = \ln(x)$.]

- This portion of the exam must be turned in **no later** than 4:30pm on Wednesday, October 16$^{\text{th}}$, 2013.
- You may use notes, textbooks, and existent online resources to complete these problem.
- You may **not** ask anyone (except me and Dr. Vicky Klima) for help.

6. (5 points) Explain why $111 \in U(997)$. Then compute 111^{-1}.

7. (9 points) Dihedral Problem.

(a) Draw a regular hexagon and label the reflective symmetries: $V_1, V_2, \ldots, V_6$ (moving around the hexagon in the counter-clockwise direction). Also, let R_{60° be the counter-clockwise rotation of 60°. Draw a few pictures to compute $R_{60^\circ} V_1$.

(b) Recall that $D_{10} = \langle x, y \mid x^{10} = 1, y^2 = 1, (xy)^2 = 1 \rangle = \{1, x, x^2, \ldots, x^9, y, xy, x^2y, \ldots, x^9y\}$.
Simplify $\alpha = x^4yx^2x^{-3}y^3xy^{-7}$.

(c) Make a table listing the orders of the elements of D_{24} as well as how many elements there are of each order.

8. (8 points) Show x and gxg^{-1} have the same order (in a finite group). Can $x = gxg^{-1}$? If so, what does this say about g and x? If not, why not?

9. (5 points) Suppose that $\sigma = (13)(24765)$ and your classmate claims that $\tau\sigma\tau^{-1} = (142)(3765)$. Is this possible? If so, what might τ be? If not, why not?

10. (10 points) Matrices.

(a) Let $H = \left\{ \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \mid a, b \in \mathbb{R} \text{ and } a, b \text{ not both } 0 \right\}$.

Using a subgroup test, show that H is a subgroup of $\mathrm{GL}_2(\mathbb{R})$.

(b) Prove that $\mathbb{C}_{\neq 0} = \{x + yi \mid x, y \in \mathbb{R} \text{ and } x, y \text{ not both } 0 \} \cong H$