Math 3110 **Final Exam** **December 7^{th}, 2015**

Name: ________________________________ **Be sure to show your work!**

1. (10 points) Working in $\mathbb{Z}_{18}$.

(a) $I = (6) = \langle 6 \rangle = \Big\{$ ______________________ $\Big\}$ and ${\mathbb{Z}_{18}}/{I} = \Big\{$ ______________________ $\Big\}$.

(b) For each element in ${\mathbb{Z}_{18}}/{I}$, state whether that element is zero, a zero divisor, a unit, or none of the above. If it is a unit, give its inverse. If it is an zero divisor, show that this is the case.

(c) It turns out that there is a ring homomorphism $\varphi : \mathbb{Z}_{18} \to \mathbb{Z}_6$ which is onto and has $I = (6) = \text{Ker}(\varphi)$. Is φ one-to-one? Explain why or why not. What does the first isomorphism theorem say in this case?

(d) Is $I = (6)$ a prime or maximal ideal in $\mathbb{Z}_{18}$? Why or why not?

2. (12 points) Groups: Isomorphic or not.

(a) Explain why $A_4 \ncong \mathbb{Z}_2 \oplus \mathbb{Z}_6$ [not isomorphic].

(b) Explain why $Q = \{\pm 1, \pm i, \pm j, \pm k\} \ncong D_4$ [not isomorphic].

(c) Explain why $U(10) \cong \mathbb{Z}_4$ [are isomorphic].

3. (8 points) Rings: Explain why each pair of **rings** are not isomorphic.

(a) $\mathbb{Z} \ncong \mathbb{E}$ where $\mathbb{E} = 2\mathbb{Z} = \{n \in \mathbb{Z} \mid n \text{ is even}\}$.

(b) $\mathbb{R}^{3\times 3} \ncong \mathbb{C}$ where $\mathbb{R}^{3\times 3}$ is the ring of 3×3 real matrices and $\mathbb{C}$ is the complex numbers.

4. (9 points) Workin' in $\mathbb{Z}_{88}$. [Note: $88 = 2^3 \cdot 11$]

(a) Fill out the following table (for $\mathbb{Z}_{88}$):

order =								
number of elements with this order =								

(b) Draw $\mathbb{Z}_{88}$'s lattice of ideals.

(c) Which ideals are prime? maximal?

5. (6 points) Workin' in $\mathbb{Z}_{323}$.

(a) Is 221 zero, a unit, a zero divisor, or none of the above in $\mathbb{Z}_{323}$? If 221 is a zero divisor, prove it. If 221 is a unit, find it's inverse.

(b) Is 20 zero, a unit, a zero divisor, or none of the above in $\mathbb{Z}_{323}$? If 20 is a zero divisor, prove it. If 20 is a unit, find it's inverse.

6. (18 points) Sub-things

(a) Let $H = \{1, y\}$. Explain why H is a subgroup of $D_5 = \{1, x, x^2, x^3, x^4, y, xy, x^2y, x^3y, x^4y\}$ (of course $x^5 = 1$, $y^2 = 1$, and $xyxy = 1$), then show H is **not** a **normal** subgroup of D_5.

(b) Let $S = 2\mathbb{Z} \oplus 3\mathbb{Z} = \{(2x, 3y) \mid x, y \in \mathbb{Z}\}$. Show S is a **subring** of $\mathbb{R} \oplus \mathbb{Q}$.

(c) Let $\mathbb{Q}[i] = \{a + bi \mid a, b \in \mathbb{Q}\}$ where $i = \sqrt{-1}$. Show that $\mathbb{Q}[i]$ is a sub**field** of the complex numbers $\mathbb{C}$.

(d) Can a subring of an integral domain fail to be an integral domain? What about a quotient of an integral domain?

7. (15 points) An ideal question.

(a) Let $\varphi : R \to S$ be a ring homomorphism. Show that $\text{Ker}(\varphi)$ is an ideal of R.

(b) Let R be a commutative ring with 1 and I an ideal of R. Show that $I = R$ if and only if I contains a unit of R.

(c) Let $\varphi : R \to S$ be a ring homomorphism where $\text{Ker}(\varphi) = \{0\}$. Show that φ is one-to-one.

8. (8 points) The Fundamental Theorem of Finite Abelian Groups.

(a) List all of the non-isomorphic abelian groups of order $100 = 2^2 \cdot 5^2$. Circle any that are cyclic.

(b) Which of the abelian groups of order 100 contain elements of order 25?

9. (7 points) Let $H = \{(1), (12)(34), (13)(24), (14)(23)\}$. It can be shown that $H \lhd A_4$. Write down a Cayley table for ${}^{A_4}\!/\!_{H}$. Is this quotient a cyclic group?

10. (7 points) Recall that if G is a group and $g \in G$, then $\varphi_g(x) = gxg^{-1}$ is called an inner automorphism. Prove that φ_g is an automorphism.