

Name: _____

Be sure to show your work!

1. (20 points) Definition and Basics

- (a) Suppose that G is a non-empty set equipped an operation. What 4 things do I need to check to see if G is a group? Give details.

1:

2:

3:

4:

What additional property needs to hold for G to be an **Abelian** group?

5:

- (b) Let $G = \mathbb{Z}_{>0}$ be the set of positive integers. Define a binary operation as follows: $x \star y = x^y$ (example: $2 \star 3 = 2^3 = 8$). G paired with this operation is **not** a group. Why? [Use concrete counterexample(s).]

- (c) Let $G = \{n\sqrt{2} \mid n \in \mathbb{Q}_{\neq 0}\}$. (The non-zero rational multiples of $\sqrt{2}$. For example: $\sqrt{2}$ and $-\frac{3\sqrt{2}}{4}$ both belong to G .) G is **not** a group under multiplication. Why? [Use concrete counterexample(s).]

2. (20 points) Some modular arithmetic.

(a) What is the inverse of 80 in the group \mathbb{Z}_{100} ? What operation makes \mathbb{Z}_{100} a group?

(b) Is 80 an element of $U(100)$? If not, why not? If so, why so and what is its inverse?

(c) Is 11 an element of $U(100)$? If not, why not? If so, why so and what is its inverse?

(d) What is the order of 80 in \mathbb{Z}_{100} ?

(e) Draw the subgroup lattice for \mathbb{Z}_{100} .

3. (25 points) More Modular Arithmetic.

(a) List the elements of $U(10)$. Then find their **orders** and the list the elements in **cyclic subgroup** generated by that element. [*Note:* There may be more spaces than you need.]

$x =$						
$ x =$						
$\langle x \rangle =$	{ }	{ }	{ }	{ }	{ }	{ }

(b) Is $U(10)$ cyclic? **Yes** / **No** (Circle the correct answer.)

(c) Find $\langle A \rangle$ in $GL_2(\mathbb{Z}_4)$ where $A = \begin{bmatrix} 3 & 0 \\ 1 & 1 \end{bmatrix}$. What is the order of A ? What is A^{-1} ?

(d) Find $7^{-1} \pmod{25}$ using the extended Euclidean algorithm [Don't just guess and check].

4. (15 points) Recall $D_n = \{1, x, \dots, x^{n-1}, y, xy, \dots, x^{n-1}y\} = \langle x, y \mid x^n = 1, y^2 = 1, (xy)^2 = 1 \rangle$.

(a) Use the relations for D_8 to simplify $x^{-2}y^3x^{11}yx^2y^{124}$

(b) What is the inverse of x^6 in D_8 ? What is its order?

(c) Fill in the Cayley table for D_3 :

	1	x	x^2	y	xy	x^2y
1	1	x	x^2	y	xy	x^2y
x	x					
x^2	x^2					
y	y					
xy	xy					
x^2y	x^2y					

(d) The **rotations** form a subgroup of D_3 . Explain why. Is this a cyclic subgroup?

5. (20 points) Proofs!

(a) Choose one of the following:

- I. Suppose that $(ab)^2 = a^2b^2$ for all $a, b \in G$. Prove that G is abelian.
- II. Let $G = \langle g \rangle$ be a cyclic group. Prove that G is abelian.

(b) Choose one of the following: (You **must** use a subgroup test in your proof.)

- I. Prove that $H = \{x \in \mathbb{Z} \mid \text{there exists } k, \ell \in \mathbb{Z} \text{ such that } x = 4k + 6\ell\}$ is a subgroup of \mathbb{Z} .
- II. Prove that $K = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \mid a, b \in \mathbb{R} \text{ and } a, b \neq 0 \right\}$ is a subgroup of $\text{GL}_2(\mathbb{R})$.