Math 3110 **Test #3** **November 6^{th}, 2015**

Name: ANSWER KEY **Be sure to show your work!**

1. (15 points) Getting things in order...

(a) Let $G = A_4 \oplus D_3$ where $A_4 = \{(1),(123),(132),(124),(142),(134),(143),(234),(243),(12)(34),(13)(24),(14)(23)\}$ and $D_3 = \langle x, y \mid x^3 = 1, y^2 = 1, xyxy = 1\rangle = \{1, x, x^2, y, xy, x^2y\}$.

The order of G is $|G| =$ $|A_4 \times D_3| = |A_4| \cdot |D_3| = 12 \cdot 6 = 72$

What is the largest element order in $A_4 \oplus D_3$? Give an example of such an element and explain why it has the largest possible order.

The orders of the elements in A_4 are 1, 2 (eg. (12)(34)), and 3 (eg. (123)). In D_3 there are elements of orders 1, 2 (the reflections), and 3 (the non-trivial rotations) as well. Recall that in a direct product group we have: $|(A,B)| = \text{lcm}(|A|,|B|)$. Thus we can have orders coming the the least common multiple of 1, 2, and 3 with 1, 2, and 3. The largest possible order is thus $\text{lcm}(2,3) = \text{lcm}(3,2) = 6$.

For example: $|(\,(12)(34), x\,)| = \text{lcm}(|(12)(34)|, |x|) = \text{lcm}(2,3) = 6$. Likewise, the order of $(\,y,(123)\,)$ also has order 6.

(b) Let G be a group with subgroups H, K, and L such that $H \subseteq L$ and $K \subseteq L$. In addition, suppose that we know $|H| = 2$, $|K| = 3$, and $|G| = 24$. What are the possible orders of L?

We have L is a subgroup of G, so by Lagrange's theorem $|L|$ divides $|G| = 24$. Thus $|L| = 1,2,3,4,6,8,12,$ or 24. However, K is a subgroup of L, so $|K| = 3$ divides $|L|$. Thus $|L|$ is a multiple of 3. This rules out $1,2,4,$ and 8. So $|L| = 3,6,12,$ or 24. Finally, H is a subgroup of L, so $|H| = 2$ divides $|L$. Thus $|L|$ is even. Therefore, $|L|$ is $\boxed{6, 12, \text{ or } 24}$.

2. (10 points) Let $G = \left\{ \begin{bmatrix} 1 & a \\ 0 & b \end{bmatrix} \;\middle|\; a, b \in \mathbb{R} \text{ and } b \neq 0 \right\}$. It turns out that G is a group under matrix multiplication. Let $H = \left\{ \begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix} \;\middle|\; c \in \mathbb{R} \right\}$. Show that H is a normal subgroup of G. As a help, notice that $\begin{bmatrix} 1 & a \\ 0 & b \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -a/b \\ 0 & 1/b \end{bmatrix}$.

[You may **not** assume that H is a subgroup – prove this as well.]

Obviously H is a non-empty subset of G. Let's check for: closure under matrix multiplication, closure under inverses, and invariance under conjugation.

- Let $A = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \in H$. Then $AB = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & a+b \\ 0 & 1 \end{bmatrix} \in H$.
- Let $A = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \in H$. Then $A^{-1} = \begin{bmatrix} 1 & -a/1 \\ 0 & 1/1 \end{bmatrix} = \begin{bmatrix} 1 & -a \\ 0 & 1 \end{bmatrix} \in H$.
- Let $Y = \begin{bmatrix} 1 & a \\ 0 & b \end{bmatrix} \in G$ and $X = \begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix} \in H$. Then $Y^{-1}XY = \begin{bmatrix} 1 & -a/b \\ 0 & 1/b \end{bmatrix}\begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & a \\ 0 & b \end{bmatrix} = \begin{bmatrix} 1 & -a/b \\ 0 & 1/b \end{bmatrix}\begin{bmatrix} 1 & a+bc \\ 0 & b \end{bmatrix}$

$$= \begin{bmatrix} 1 & a+bc-(a/b)b \\ 0 & (1/b)b \end{bmatrix} = \begin{bmatrix} 1 & bc \\ 0 & 1 \end{bmatrix} \in H.$$

Therefore, $H \triangleleft G$.

3. (15 points) A previous homework set showed that $H = \{1, x^3, x^6\}$ is a normal subgroup of $D_9 = \langle x, y \mid x^9 = 1, y^2 = 1, xyxy = 1\rangle = \{1, x, \dots, x^8, y, xy, \dots, x^8y\}$.

(a) Quick questions about D_9/H. The order of D_9/H is $\dfrac{|D_9|}{|H|} = \dfrac{18}{3} = 6$.

The identity of D_9/H is $1H = H = \{1, x^3, x^6\}$. $(x^5H)^{-1} =$ $(x^5)^{-1}H = x^4H = \{x, x^4, x^7\}$.

The order of xH in D_9/H is 3. The size of the set xH is 3.

Scratch work: Notice that $(xH)^1 = xH$, $(xH)^2 = x^2H$, and $(xH)^3 = x^3H = H$ (since $x^3 \in H$), so the order of xH (as an element of the quotient group D_9/H) is 3. Next, $xH = x\{1, x^3, x^6\} = \{x, x^4, x^7\}$, so the size of the set xH is also 3.

(b) This theorem should be helpful: Every group of order $2p$ (where p is an odd prime) is either cyclic (isomorphic to $\mathbb{Z}_{2p}$) or dihedral (isomorphic to D_p).

$xH\,yH = xyH = xy\{1, x^3, x^6\} = \{xy, xyx^3, xyx^6\} = \{xy, x^4y, x^7y\}$ [List the elements in the resulting coset.]

$yH\,xH = yxH = yx\{1, x^3, x^6\} = \{yx, yx^4, yx^7\} = \{x^2y, x^5y, x^8y\}$ [List the elements in the resulting coset.]

Therefore, $xHyH \neq yHxH$. This means that D_9/H is not abelian. So D_9/H is a non-abelian group of order $6 = 2 \cdot 3$. By the above theorem, it must be isomorphic to D_3.

The above calculation tells us that D_9/H is isomorphic to $\underline{\quad D_3 \quad}$.

4. (10 points) Consider $\mathbb{Z}_{12}/H$ where $H = \langle 3 \rangle = \{0, 3, 6, 9\}$. List all of the cosets (and their contents) of H in $\mathbb{Z}_{12}$. Then make a Cayley table for this quotient group.

Keep in mind the operation in $\mathbb{Z}_{12}$ is addition modulo 12, so we will have "additive" cosets. The order of our quotient group is $\dfrac{|\mathbb{Z}_{12}|}{|H|} = \dfrac{12}{4} = 3$. Thus we should find 3 distinct cosets.

The cosets: $\boxed{0 + H = H = \{0, 3, 6, 9\}, \quad 1 + H = \{1+0, 1+3, 1+6, 1+9\} = \{1, 4, 7, 10\}, \quad \text{and} \quad 2 + H = \{2, 5, 8, 11\}}$.

Next, we need our Cayley table. Most of the computations are completely mundane such as: $H + (2 + H) = (0 + H) + (2 + H) = (0 + 2) + H = 2 + H$ and $(1 + H) + (1 + H) = (1 + 1) + H = 2 + H$. Some are *slightly* more interesting such as $(1 + H) + (2 + H) = (1 + 2) + H = 3 + H = H$ since $3 \in H$. Also, $(2 + H) + (2 + H) = (2 + 2) + H = 4 + H = 1 + H$ since $4 \in 1 + H$.

	H	$1 + H$	$2 + H$
H	H	$1 + H$	$2 + H$
$1 + H$	$1 + H$	$2 + H$	H
$2 + H$	$2 + H$	H	$1 + H$

Of course, we immediately recognize that this is the Cayley table of $\mathbb{Z}_3$ which makes sense since $\mathbb{Z}_{12}/H = \mathbb{Z}_{12}/\langle 3 \rangle \cong \mathbb{Z}_3$.

5. (15 points) Let $\varphi : \mathbb{Z}_8 \to \mathbb{Z}_{12}$ be defined by $\varphi(x) = 3x$.

(a) Show that φ is a homomorphism. [Do we need to prove that φ is well-defined?]

Since φ is defined in terms of a representative "x" of an equivalence class (i.e. all integers congruent to x modulo 8), we **do** need to check to see if φ is well-defined.

Suppose $x = y \pmod 8$. Then $x = y + 8k$ for some $k \in \mathbb{Z}$. Therefore, $3x = 3(y + 8k) = 3y + 24k = 3y + 12(2k)$. This means that $3x = 3y \pmod{12}$. Therefore, $\varphi(x) = \varphi(y) \pmod{12}$ and so φ is well-defined.

Next, notice that for all $a, b \in \mathbb{Z}_8$, $\varphi(a + b) = 3(a + b) = 3a + 3b = \varphi(a) + \varphi(b)$. Thus φ is operation preserving and so φ is a homomorphism.

(b) Compute the kernel and image of φ.

Let's record what φ does (keep in mind that the outputs are computed "mod 12"): $0 \mapsto 3 \cdot 0 = 0$, $1 \mapsto 3 \cdot 1 = 3$, $2 \mapsto 3 \cdot 2 = 6$, $3 \mapsto 3 \cdot 3 = 9$, $4 \mapsto 3 \cdot 4 = 12 = 0$, $5 \mapsto 3 \cdot 5 = 15 = 3$, $6 \mapsto 3 \cdot 6 = 18 = 6$, and $7 \mapsto 3 \cdot 7 = 21 = 9$.

$\boxed{\ker(\varphi) = \{x \in \mathbb{Z}_8 \mid \varphi(x) = 0\} = \{0, 4\} = \langle 4 \rangle}$ (in $\mathbb{Z}_8$) and $\boxed{\text{Im}(\varphi) = \varphi(\mathbb{Z}_8) = \{0, 3, 6, 9\} = \langle 3 \rangle}$ (in $\mathbb{Z}_{12}$).

(c) When applied to this φ, what does the first isomorphism theorem tell us?

$$\mathbb{Z}_8/\ker(\varphi) \cong \text{Im}(\varphi) \qquad \Longrightarrow \qquad \mathbb{Z}_8/\langle 4 \rangle_{\text{(in } \mathbb{Z}_8)} \cong \langle 3 \rangle_{\text{(in } \mathbb{Z}_{12})}$$

6 (10 points) Let $\varphi : \mathrm{GL}_2(\mathbb{R}) \to \mathbb{R}_{\neq 0}$ be defined by $\varphi(A) = \det(A)$. Show φ is a homomorphism. What does the first isomorphism theorem tell us in this particular situation?

There's nothing special about $n = 2$, so I'll work with $\mathrm{GL}_n(\mathbb{R})$ for any $n \in \mathbb{Z}_{>0}$. Let $A, B \in \mathrm{GL}_n(\mathbb{R})$. Then $\varphi(AB) = \det(AB) = \det(A)\det(B) = \varphi(A)\varphi(B)$. Thus φ is a homomorphism. [Of course, this assumes a basic, yet not that easy to prove, property of determinants.]

$\ker(\varphi) = \{A \in \mathrm{GL}_n(\mathbb{R}) \mid \varphi(A) = 1\} = \{A \in \mathrm{GL}_n \mid \det(A) = 1\} = \mathrm{SL}_n(\mathbb{R})$ (the "special linear group"). By the way, this shows that $\mathrm{SL}_n(\mathbb{R}) \lhd \mathrm{GL}_n(\mathbb{R})$ (since kernels are normal).

Notice that φ is onto. Consider $r \in \mathbb{R}_{\neq 0}$. Then let $A = \begin{bmatrix} r & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$. Then $\det(A) = r \cdot 1 \cdots 1 = r \neq 0$ so $A \in \mathrm{GL}_n(\mathbb{R})$ and $\varphi(A) = \det(A) = r$. Thus $r \in \mathrm{Im}(\varphi)$. So in our situation the first isomorphism theorem says...

$$\mathrm{GL}_n(\mathbb{R}) / \ker(\varphi) \cong \mathrm{Im}(\varphi) \qquad \Longrightarrow \qquad \mathrm{GL}_n(\mathbb{R}) / \mathrm{SL}_n(\mathbb{R}) \cong \mathbb{R}_{\neq 0}$$

7. (25 points) Finite Abelian Groups

(a) List all of the non-isomorphic abelian groups of order $225 = 3^2 5^2$. Circle any that are cyclic.

There are 2 ways to split up 3^2: $3^2 = 9$ or $3 \cdot 3$. The same is true for 5^2: $5^2 = 25$ or $5 \cdot 5$. This leaves us with $2 \cdot 2 = 4$ isomorphism classes:

- $\boxed{\mathbb{Z}_9 \oplus \mathbb{Z}_{25} \cong \mathbb{Z}_{225}}$ [These are cyclic.]
- $\mathbb{Z}_9 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_5 \cong \mathbb{Z}_5 \oplus \mathbb{Z}_{45}$
- $\mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_{25} \cong \mathbb{Z}_3 \oplus \mathbb{Z}_{75}$
- $\mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_5 \cong \mathbb{Z}_{15} \oplus \mathbb{Z}_{15}$

(b) How many non-isomorphic abelian groups of order 14,346,832,500 are there?

Note: $14{,}346{,}832{,}500 = 2^2 \cdot 3^2 \cdot 5^4 \cdot 7^3 \cdot 11 \cdot 13^2$ and there are 5 non-isomorphic abelian groups of order $625 = 5^4$. ☺

There are 2 ways to break up 2^2. The same is true for 3^2 and 13^2. There are 3 ways to break up 7^3. 11 can't be split up. Finally, there are 5 ways to split up 5^4 (as alluded to in the note above). This means that there are $p(2)p(2)p(4)p(3)p(1)p(2) = 2 \cdot 2 \cdot 5 \cdot 3 \cdot 1 \cdot 2 =$ $\boxed{\text{120 non-isomorphic abelian groups of order 14,346,832,500}}$.

(c) Are the groups $\mathbb{Z}_{10} \oplus \mathbb{Z}_{20} \oplus \mathbb{Z}_{30}$ and $\mathbb{Z}_{100} \oplus \mathbb{Z}_{60}$ isomorphic? Explain your answer.

Although not the quickest solution, we can determine if these groups are isomorphic by breaking them up as much as possible (keeping in mind that $\mathbb{Z}_{k\ell} \cong \mathbb{Z}_k \oplus \mathbb{Z}_\ell$ if and only if k and ℓ are relatively prime).

$$\mathbb{Z}_{10} \oplus \mathbb{Z}_{20} \oplus \mathbb{Z}_{30} \cong \mathbb{Z}_2 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_5$$

vs.

$$\mathbb{Z}_{100} \oplus \mathbb{Z}_{60} \cong \mathbb{Z}_4 \oplus \mathbb{Z}_{25} \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5 \cong \mathbb{Z}_4 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_{25}$$

Answer: $\boxed{\textbf{No}}$. Notice that we cannot put $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ together to get $\mathbb{Z}_4$ and also we cannot put $\mathbb{Z}_5 \oplus \mathbb{Z}_5$ to get $\mathbb{Z}_{25}$. So these groups are not isomorphic.

(d) Is the group $\mathbb{Z}_{14} \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_{81}$ cyclic? Explain you answer.

Answer: $\boxed{\textbf{Yes}}$. Notice that $14 = 2 \cdot 7$, $5 = 5$, and $81 = 3^4$. These numbers are pairwise relatively prime, therefore all of these groups can be combined: $\mathbb{Z}_{14} \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_{81} \cong \mathbb{Z}_{14 \cdot 5 \cdot 81} = \mathbb{Z}_{5{,}670}$ is cyclic.