Math 3110 # Test #3 **November 6^{th}, 2015**

Name: ______________________________ **Be sure to show your work!**

1. (15 points) Getting things in order...

(a) Let $G = A_4 \oplus D_3$ where $A_4 = \{(1), (123), (132), (124), (142), (134), (143), (234), (243), (12)(34), (13)(24), (14)(23)\}$ and $D_3 = \langle x, y \mid x^3 = 1, y^2 = 1, xyxy = 1 \rangle = \{1, x, x^2, y, xy, x^2y\}$.

The order of G is $|G| =$ ____________

What is the largest element order in $A_4 \oplus D_3$? Give an example of such an element and explain why it has the largest possible order.

(b) Let G be a group with subgroups H, K, and L such that $H \subseteq L$ and $K \subseteq L$. In addition, suppose that we know $|H| = 2$, $|K| = 3$, and $|G| = 24$. What are the possible orders of L?

2. (10 points) Let $G = \left\{ \begin{bmatrix} 1 & a \\ 0 & b \end{bmatrix} \;\middle|\; a, b \in \mathbb{R} \text{ and } b \neq 0 \right\}$. It turns out that G is a group under matrix multiplication. Let $H = \left\{ \begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix} \;\middle|\; c \in \mathbb{R} \right\}$. Show that H is a normal subgroup of G. As a help, notice that $\begin{bmatrix} 1 & a \\ 0 & b \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -a/b \\ 0 & 1/b \end{bmatrix}$.

[You may **not** assume that H is a subgroup – prove this as well.]

3. (15 points) A previous homework set showed that $H = \{1, x^3, x^6\}$ is a normal subgroup of $D_9 = \langle x, y \mid x^9 = 1, y^2 = 1, xyxy = 1\rangle = \{1, x, \ldots, x^8, y, xy, \ldots, x^8y\}$.

(a) Quick questions about ${D_9}/{H}$.

The order of ${D_9}/{H}$ is ______________________________.

The identity of ${D_9}/{H}$ is _______________. $(x^5H)^{-1} =$ ______________________________.

The order of xH in ${D_9}/{H}$ is _______________. The size of the set xH is _______________.

Scratch work:

(b) This theorem should be helpful: Every group of order $2p$ (where p is an odd prime) is either cyclic (isomorphic to $\mathbb{Z}_{2p}$) or dihedral (isomorphic to D_p).

$xH\,yH =$ ______$H = \{ \qquad \}$ [List the elements in the resulting coset.]

$yH\,xH =$ ______$H = \{ \qquad \}$ [List the elements in the resulting coset.]

The above calculation tells us that ${D_9}/{H}$ is isomorphic to ______________________________.

4. (10 points) Consider ${\mathbb{Z}_{12}}/{H}$ where $H = \langle 3\rangle = \{0, 3, 6, 9\}$. List all of the cosets (and their contents) of H in $\mathbb{Z}_{12}$. Then make a Cayley table for this quotient group.

5. (15 points) Let $\varphi : \mathbb{Z}_8 \to \mathbb{Z}_{12}$ be defined by $\varphi(x) = 3x$.

(a) Show that φ is a homomorphism. [Do we need to prove that φ is well-defined?]

(b) Compute the kernel and image of φ.

(c) When applied to this φ, what does the first isomorphism theorem tell us?

6 (10 points) Let $\varphi : \mathrm{GL}_2(\mathbb{R}) \to \mathbb{R}_{\neq 0}$ be defined by $\varphi(A) = \det(A)$. Show φ is a homomorphism. What does the first isomorphism theorem tell us in this particular situation?

7. (25 points) Finite Abelian Groups

(a) List all of the non-isomorphic abelian groups of order $225 = 3^2 5^2$. Circle any that are cyclic.

(b) How many non-isomorphic abelian groups of order 14,346,832,500 are there?

Note: $14{,}346{,}832{,}500 = 2^2 \cdot 3^2 \cdot 5^4 \cdot 7^3 \cdot 11 \cdot 13^2$ and there are 5 non-isomorphic abelian groups of order $625 = 5^4$. ☺

(c) Are the groups $\mathbb{Z}_{10} \oplus \mathbb{Z}_{20} \oplus \mathbb{Z}_{30}$ and $\mathbb{Z}_{100} \oplus \mathbb{Z}_{60}$ isomorphic? Explain your answer.

(d) Is the group $\mathbb{Z}_{14} \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_{81}$ cyclic? Explain you answer.