Math 3110 # Test #3 **November 2^{nd}, 2020**

Name: <u>Answer Key</u> **Be sure to show your work!**

1. (15 points) Getting things in order...

(a) Let $G = D_5 \times S_4$ where $D_5 = \langle x, y \mid x^5 = 1, y^2 = 1, xyxy = 1 \rangle$ and S_4 is permutations on 4 things.

The order of G is $|G| =$ $|D_5| \cdot |S_4| = 5(2) \cdot 4! = 10 \cdot 24 = \boxed{240}$

What is the largest element order in $D_5 \times S_4$? Give an example of such an element and explain why it has the largest possible order.

The elements in D_5 have orders $1, 5$ (identity and rotations like x) and 2 (reflections). The elements in S_4 have orders $1, 2, 3$, and 4 (e.g., the identity, transpositions or stuff like (12)(34), 3-cycles, and 4-cycles like (1234)). Recall that $|(a, b)| = \text{lcm}(|a|, |b|)$ so thinking of various lcm's, the largest we can get is $\text{lcm}(5, 4) = 20$.

The largest element order in $D_5 \times S_4$ is $\boxed{|(x, (1234))| = \text{lcm}(|x|, |(1234)|) = \text{lcm}(5, 4) = 20}$.

(b) Let G be a group of order 36 with subgroup H of order 12. Let K be some other subgroup of G.

Can $H \cap K$ have order 9? Why or why not?

$\boxed{\text{No.}}$ The order of $H \cap K$ must divide both the order of H and the order of K. But 9 does not divide $12 = |H|$.

If $H \cap K$ has order 6, what are the possible orders of K?

From Lagrange's theorem, we have that the order of K must divide the order of G (i.e., $|K|$ divides 36), but also since $H \cap K$ is a subgroup of K, we must have that $|H \cap K| = 6$ divides the order of K. Therefore, $|K|$ is a divisor of 36 and a multiple of 6: $\boxed{|K| = 6, 12, 18, \text{ or } \cancel{36}}$.

(Mostly) Ridiculous Notes: We can rule out 36 since that would require $K = G$ and then $H \cap K = H \cap G = H$ would be of order 12 (not 6).

In fact, much more can be said. It turns out that the only orders that actually occur in this situation are $|K| = 6, 12$, and 18. But this is way beyond what I was looking for here. The following GAP code cycles through all groups of order 36 (up to isomorphism) and looks at all subgroups of order 12 intersected with subgroups K such that the intersection has order 6. It then prints out orders of such K's:

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for n in [1..14] do
   G := SmallGroup(36,n);
   L := AllSubgroups(G);

   L12:=[];
   for H in L do
      if Size(H)=12 then Append(L12,[H]); fi;
   od;

   for H in L12 do
       for K in L do
           if Size(Intersection(H,K))=6 then
                Print(Size(K)); Print(", ");
         fi;
       od;
   od;
od;
```

2. (10 points) Let H and K be normal subgroups of G. Prove that $H \cap K$ is a normal subgroup of G.

First, notice that $H \cap K$ is a subset of G since both H and K are subsets of G. Both H and K contain the identity 1 of G since they are subgroups of G. Therefore, $1 \in H \cap K$. Thus $H \cap K$ is a non-empty subset of G.

Next, suppose $a, b \in H \cap K$. Then $a, b \in H$ and $a, b \in K$. Therefore, $ab \in H$ (since H is a subgroup, it is closed under the operation). Likewise, $ab \in K$. Thus since $ab \in H$ and $ab \in K$, we have $ab \in H \cap K$.

Also, given $a \in H \cap K$ so that $a \in H$ and $a \in K$, we have $a^{-1} \in H$ and $a^{-1} \in K$ (H and K are subgroups and thus closed under inverses). Therefore, $a^{-1} \in H \cap K$. We have now established that $H \cap K$ is a subgroup of G.

Finally, let $g \in G$ and $a \in H \cap K$. Then $a \in H$ and since H is normal, we have $gag^{-1} \in H$ (closed under conjugation). Likewise, $gag^{-1} \in K$. Therefore, $gag^{-1} \in H \cap K$. Thus $H \cap K \triangleleft G$.

3. (15 points) Consider $H = \{1, x^2\}$ in $D_4 = \langle x, y \mid x^4 = 1, y^2 = 1, xyxy = 1 \rangle = \{1, x, x^2, x^3, y, xy, x^2y, x^3y\}$. Note that H is a normal subgroup of D_4.

(a) Quick questions about D_4/H.

The order of D_4/H is $|D_4|/|H| = 8/2 = \boxed{4}$.

The identity of D_4/H is $1H = \boxed{H}$. $(xH)^{-1} = x^{-1}H = x^3H = \boxed{xH}$.

The order of xH in D_4/H is $\boxed{2}$ since xH is its own inverse. The size of the set xH is the same as H: $\boxed{2}$.

A few notes: The order of a quotient is just the index of the subgroup. By Lagrange, this is the order of the group divided by the order of the subgroup. Next, the identity of any quotient group is just the subgroup you are quotienting by. To find an inverse, you take the inverse of the representative. Notice that $xH = \{x \cdot 1, x \cdot x^2\} = \{x, x^3\} = x^3H$. Also, since $(xH)^{-1} = xH$ (and xH isn't the identity) it must have order 2. Alternatively, we compute: $xH \neq H$ and $(xH)^2 = x^2H = H$. Thus order 2. Finally, all of the cosets have the same size. This is the same as the size of the subgroup itself. In our case, that's 2.

(b) (Still referring to part (a).) $H = Z(D_4)$ (our H above is actually the center of D_4). Is D_4/H cyclic? Abelian? Identify this quotient group (it has a special name).

By the G/Z-Theorem: $G/Z(G)$ is cyclic if and only if G is Abelian. Since D_4 is not Abelian, $D_4/H = D_4/Z(D_4)$ must not be cyclic. However, notice that $\left| D_4/H \right| = 4$. We know that there are only two groups of order 4 (up to isomorphism): the cyclic group of order 4 and the Klein 4-group. Therefore, $\boxed{D_4/H \text{ is not cyclic, but it is Abelian. It is the Klein 4-group.}}$

4. (10 points) Consider $\mathbb{Z}_{20}/H$ where $H = \langle 4 \rangle = \{0, 4, 8, 12, 16\}$. List all of the cosets (and their contents) of H in $\mathbb{Z}_{20}$. Then make a Cayley table for this quotient group.

Cosets: $H = \{0, 4, 8, 12, 16\}$, $1 + H = \{1, 5, 9, 13, 17\}$, $2 + H = \{2, 6, 10, 14, 18\}$, and $3 + H = \{3, 7, 11, 15, 19\}$.

$\mathbb{Z}_{20}/H$	H	$1+H$	$2+H$	$3+H$
H	H	$1+H$	$2+H$	$3+H$
$1+H$	$1+H$	$2+H$	$3+H$	H
$2+H$	$2+H$	$3+H$	H	$1+H$
$3+H$	$3+H$	H	$1+H$	$2+H$

What is the order of $2 + H$ in $\mathbb{Z}_{20}/H$?

$2 + H \neq H$, $(2 + H) + (2 + H) = (2 + 2) + H = 4 + H = H$. Thus $2 + H$ has $\boxed{\text{order } 2}$. Alternatively, we could just note that $2 + H$ is not the identity but it is its own inverse (as evidenced in the above Cayley table). Thus its order is 2.

5. (15 points) Something is terribly, horribly wrong!

(a) Let $H \triangleleft S_4$. Why is $S_4 \diagup H \cong \mathbb{Z}_{18}$ impossible?

If there were such a quotient group, its order would be 18. However, the order of S_4/H must divide the order of the group (since the order of the quotient times the order of H must be the order of S_4) and $|S_4| = 4! = 24$. But $\boxed{\text{18 does not divide 24}}$. Therefore, there can be no such quotient of S_4.

(b) Let $\varphi : S_4 \to G$ be a homomorpihsm into some group G. Let $H = \langle (12) \rangle = \{(1), (12)\}$. Why is $\ker(\varphi) = H$ impossible?

Notice that $(13)H = \{(13)(1), (13)(12)\} = \{(13), (123)\}$ whereas $H(13) = \{(1)(13), (12)(13)\} = \{(13), (132)\}$. Therefore, since $(13)H \neq H(13)$, $\boxed{H \text{ is not a normal subgroup}}$ of S_4. Therefore, H cannot be a kernel (kernels are always normal subgroups).

(c) Why can't I have an epimorphism (i.e., onto homomorphism) $\psi : D_4 \times Q \to \mathbb{Z}_{15}$ (where Q is the quaternion group)?

As a consequence of the first isomorphism theorem, we had that the order of the domain of a homomorphism is equal to the order of its kernel times the order of its range (=image). Notice that the order of the domain of ψ is $|D_4 \times Q| = |D_4| \cdot |Q| = 8 \cdot 8 = 64$ whereas the order of the image (since ψ is onto) is $|\mathbb{Z}_{15}| = 15$. But $\boxed{\text{15 does not divide 64}}$ (the order of the kernel would have to be 64/15 – which is absurd).

Note: Actually, much more can be said. Suppose $\psi : D_4 \times Q \to \mathbb{Z}_{15}$ is a homomorphism. Then the order of the range must divide both $|D_4 \times Q| = 64$ and $|\mathbb{Z}_{15}| = 15$ (since the range is a subgroup of the codomain). Thus the order of the range must divide $\gcd(64, 15) = 1$. This means that the *only* homomorphism between $D_4 \times Q$ and $\mathbb{Z}_{15}$ is the trivial morphism: $\psi(x) = 0$ for all $x \in D_4 \times Q$. In other words, ψ can't come anywhere close to being onto!

6 (10 points) Let H and K be groups. Define $\pi : H \times K \to K$ by $\pi((h, k)) = k$. Show π is a homomorphism. What does the first isomorphism theorem tell us in this particular situation?

Suppose $(a, b), (x, y) \in H \times K$. Then $\pi((a, b)(x, y)) = \pi((ax, by)) = by = \pi((a, b))\pi((x, y))$. Therefore, π is a homomorphism.

Notice that π is onto since given any $k \in K$ we have $\pi((1_H, k)) = k$ (where 1_H is the identity of H). Next,

$$\ker(\pi) = \{(h, k) \in H \times K \mid \pi((h, k)) = 1_K\} = \{(h, k) \in H \times K \mid k = 1_K\} = \{(h, 1_K) \mid h \in H\} = H \times \{1_K\}$$

(where 1_K is the identity of K). Therefore, by the first isomorphism theorem: $H \times K \diagup \ker(\pi) \cong \text{im}(\pi)$. In particular,

$$\boxed{H \times K \diagup H \times \{1_K\} \cong K}.$$

7. (25 points) Finite Abelian Groups

(a) List all of the non-isomorphic abelian groups of order $72 = 2^3 3^2$. Circle any that are cyclic.

$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_9$, $\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_3$, $\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_9$, $\mathbb{Z}_8 \times \mathbb{Z}_3 \times \mathbb{Z}_3$, and $\boxed{\mathbb{Z}_8 \times \mathbb{Z}_9}$

(b) How many non-isomorphic abelian groups of order 55,635,800 are there?

Note: $55{,}635{,}800 = 2^3 \cdot 5^2 \cdot 11^4 \cdot 19$ and there are 5 non-isomorphic abelian groups of order $14{,}641 = 11^4$. ☺

There are 3 ways to split up 2^3, 2 ways to split of 5^2, 5 ways to split of 11^4 and nothing to bust up 19^1. Therefore, there are $3 \cdot 2 \cdot 5 = \boxed{30}$ Abelian groups of order 55,635,800 (up to isomorphism).

(c) Are the groups $\mathbb{Z}_{10} \times \mathbb{Z}_{25} \times \mathbb{Z}_{36}$ and $\mathbb{Z}_{100} \times \mathbb{Z}_{90}$ isomorphic? Explain your answer.

$\boxed{\text{Yes.}}$ We can split/combine if moduli are relatively prime: $\mathbb{Z}_{10} \times \mathbb{Z}_{25} \times \mathbb{Z}_{36} \cong \mathbb{Z}_{10} \times \mathbb{Z}_{25} \times \mathbb{Z}_4 \times \mathbb{Z}_9 \cong \mathbb{Z}_{100} \times \mathbb{Z}_{90}$

(d) What is the largest order among elements of $\mathbb{Z}_{35} \times \mathbb{Z}_{10} \times \mathbb{Z}_{14}$? Explain you answer.

The biggest order we can get is the least common multiple of the largest orders in our three component groups: $\text{lcm}(35, 10, 14) = \text{lcm}(5 \cdot 7, 2 \cdot 5, 2 \cdot 7) = 2 \cdot 5 \cdot 7 = \boxed{70}$.