

Name: _____
statements. **Be sure to show your work!**

Don't merely state answers, prove your

1. (_____/15 points) Define a binary operation “ \star ” on \mathbb{Z} , as follows $x \star y = -xy$ (for all $x, y \in \mathbb{Z}$).
So, for example, $2 \star (-3) = -2(-3) = 6$.
- (a) Is $\mathbb{Z}_{>0}$ (the set of positive integers) closed with respect to \star ? Justify your answer.

(b) Is \star is an associative? Why or why not?

(c) Show that -1 is an identity for \star .

2. (_____/14 points) Consider \mathbb{Z} with some relation R . Also, let $a, b \in \mathbb{Z}$.

(a) Let aRb if and only if $a \leq b$. R is **not** an equivalence relation. Why?

(b) Let aRb if and only if $a - b$ is divisible by 2. Then R is an equivalence relation.
Describe the equivalence classes.

3. (____/15 points) Let $f : A \rightarrow A$ and $g : A \rightarrow A$ for some (non-empty) set A .

(a) Define “ f is onto”.

(b) Suppose that f and g are both one-to-one. Show that $f \circ g$ is one-to-one.

(c) Let $h : \mathbb{Z} \rightarrow \mathbb{Z}$ where $h(x) = 2x + 1$. Is h one-to-one? Is h onto?

4. (____/14 points) Prove **one** of the following (choose I. or II.):

I. Let U be a set, $A \subseteq U$, and $B \subseteq U$. Show that $(A \cap B)' = A' \cup B'$.
(Recall $X' = U - X$ is the complement of X in U .)

II. $1 + 3 + 5 + \cdots + (2n - 1) = n^2$ for all $n \geq 1$.

5. (____/14 points) Division.

(a) Let $a, b \in \mathbb{Z}$. Define $a \mid b$.

(b) Let $a, b, c \in \mathbb{Z}$. Prove that if $a \mid b$ and $b \mid c$, then $a \mid c$.

6. (____/14 points) The Euclidean Algorithm and Congruences.

(a) Use the Euclidean algorithm to find $(20, 12) = d$. Then use the Euclidean algorithm to find $x, y \in \mathbb{Z}$ such that $12x + 20y = d$.

(b) Consider the equation: $12x \equiv 3 \pmod{20}$. If possible, find a solution x such that $0 \leq x < 20$. Otherwise, explain why there is no such solution.

(c) Consider the equation: $12x \equiv 8 \pmod{20}$. If possible, find a solution x such that $0 \leq x < 20$. Otherwise, explain why there is no such solution.

7. (____/14 points) Workin' mod 4.

(a) Fill out the following tables (don't worry about brackets for equivalence classes.)

$(\mathbb{Z}_4, +)$	0	1	2	3
0				
1				
2				
3				

\mathbb{Z}_4 Addition Table

(\mathbb{Z}_4, \times)	0	1	2	3
0				
1				
2				
3				

\mathbb{Z}_4 Multiplication Table

(b) What is $-1 \pmod{4}$?

(c) What is $3^{-1} \pmod{4}$?

(d) Compute $3^{-1}(3 + 2) - 1 \pmod{4}$.