

1. (20 points) Isomorphic or not?

- (a) Is
- $\text{GL}_2(\mathbb{R}) \cong \mathbb{Z}_{100}$
- ? Why or why not?

Notice that $|\text{GL}_2(\mathbb{R})| = \infty \neq 100 = |\mathbb{Z}_{100}|$ so they are not isomorphic because their orders don't match. Alternatively, they cannot be isomorphic because $\text{GL}_2(\mathbb{R})$ is non-Abelian whereas \mathbb{Z}_{100} is Abelian. Of course, there are many other ways to see that these groups aren't isomorphic.

Answer: $\text{GL}_2(\mathbb{R})$ and \mathbb{Z}_{100} are **not** isomorphic.

- (b) Recall
- $A_4 = \{(1), (123), (132), (124), (142), (134), (143), (234), (243), (12)(34), (13)(24), (14)(23)\}$
- . Show
- $A_4 \not\cong D_6$
- (they are not isomorphic).
- Hint:*
- Think about orders of elements.

$|A_4| = 4!/2 = 12 = |D_6|$ so both groups have the same size. Both are non-Abelian, so that doesn't help either. However, notice that A_4 has elements of order 1, 2, and 3 but no elements of order 6 whereas D_6 has a rotation of order 6 (R_α with $\alpha = 60^\circ$). Now since isomorphisms send elements of order k to elements of order k , we know that isomorphic groups must have the same number of elements of the same order. So A_4 and D_6 cannot be isomorphic (because the number of elements of order 6 – none vs. some – don't match).

As an additional example, notice the quaternion group $Q = \{\pm 1, \pm i, \pm j, \pm k\}$ is not isomorphic to D_4 . Both are non-Abelian groups of order 8, and both only have elements of orders 1, 2, and 4. However, Q has only 1 element of order 2 (-1 has order 2) whereas D_4 has 5 elements of order 2 (the 180° rotation and 4 reflections). Therefore, $Q \not\cong D_4$.

- (c)
- G
- is a group of order 5. Is
- $G \cong \mathbb{Z}_5$
- ? Yes, no, maybe? Explain your answer.

Answer: Yes. 5 is a prime, groups of prime order are cyclic, and cyclic groups of the same order are isomorphic.

2. (20 points) Homomorphisms

- (a) Let
- $G = \left\{ \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \mid a \in \mathbb{R} \right\}$
- and let
- $\varphi : G \rightarrow \mathbb{R}$
- be defined by
- $\varphi \left(\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \right) = a$
- . Show that
- φ
- is a homomorphism and find the kernel of
- φ
- . Is
- φ
- 1-1? onto? an isomorphism?

Remember that the operation making G a group is matrix multiplication and the operation which makes \mathbb{R} a group is addition, so we need $\varphi(AB) = \varphi(A) + \varphi(B)$ where A and B are matrices in G .

$$\varphi \left(\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \right) = \varphi \left(\begin{bmatrix} 1 & a+b \\ 0 & 1 \end{bmatrix} \right) = a+b = \varphi \left(\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \right) + \varphi \left(\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \right)$$

Therefore, φ is a homomorphism. Now let's find the kernel of φ . Keep in mind the identity of \mathbb{R} is 0 (under addition).

$$\text{Ker}(\varphi) = \left\{ \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \mid \varphi \left(\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \right) = a = 0 \right\} = \left\{ \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \mid a = 0 \right\} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

The only element in the kernel is the identity (matrix) so φ is one-to-one. Pick $y \in \mathbb{R}$, then $\begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix} \in G$

and $\varphi \left(\begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix} \right) = y$. Therefore, φ is onto.

Since φ is a bijective homomorphism, φ is an isomorphism ($G \cong \mathbb{R}$).

- (b) Let $\psi : \mathbb{Z}_9 \rightarrow \mathbb{Z}_3$ be defined by $\psi(n) = n$. Is ψ a *well defined* homomorphism? If so, find $\text{Ker}(\psi)$. Is ψ 1-1? onto? an isomorphism?

Suppose $a \equiv b \pmod{9}$ then 9 divides $b - a$. Therefore 3 divides $b - a$. Thus $a \equiv b \pmod{3}$. So $a = b$ in \mathbb{Z}_9 implies $\psi(a) = a = b = \psi(b)$ in \mathbb{Z}_3 . Therefore, ψ is well-defined (equal things map to equal things). Next, $\psi(k + \ell) = k + \ell = \psi(k) + \psi(\ell)$ so ψ is a homomorphism.

$$\text{Ker}(\psi) = \{x \in \mathbb{Z}_9 \mid \psi(x) = x = 0 \pmod{3}\} = \{0, 3, 6\}$$

Since the kernel contains non-identity elements, ψ is not one-to-one and thus not an isomorphism (*Note:* since the kernel contains 3 elements ψ is a three-to-one map). ψ is onto since 0, 1, 2 map to themselves.

- 3. (20 points)** Write down the left multiplication operators for D_3 and translate them into permutations (in S_6) labeling the elements of D_3 as follows: $1 \rightsquigarrow 1, a \rightsquigarrow 2, a^2 \rightsquigarrow 3, b \rightsquigarrow 4, ab \rightsquigarrow 5, a^2b \rightsquigarrow 6$. Write down a subgroup of S_6 which isomorphic to D_3 (using Cayley's theorem).

Let $g, x \in S_3$. We write $L_g(x) = gx$ (so that L_g is the left multiplication by g operator). We have (for each $g \in D_3$) L_g 's action on D_3 and then this action translated to the proper labels 1, ..., 6:

$L_1 :$	1	\mapsto	1	1	\mapsto	1
	a	\mapsto	a	2	\mapsto	2
	a^2	\mapsto	a^2	3	\mapsto	3
	b	\mapsto	b	4	\mapsto	4
	ab	\mapsto	ab	5	\mapsto	5
	a^2b	\mapsto	a^2b	6	\mapsto	6
$L_a :$	1	\mapsto	a	1	\mapsto	2
	a	\mapsto	a^2	2	\mapsto	3
	a^2	\mapsto	1	3	\mapsto	1
	b	\mapsto	ab	4	\mapsto	5
	ab	\mapsto	a^2b	5	\mapsto	6
	a^2b	\mapsto	b	6	\mapsto	4
$L_{a^2} :$	1	\mapsto	a^2	1	\mapsto	3
	a	\mapsto	1	2	\mapsto	1
	a^2	\mapsto	a	3	\mapsto	2
	b	\mapsto	a^2b	4	\mapsto	6
	ab	\mapsto	b	5	\mapsto	4
	a^2b	\mapsto	ab	6	\mapsto	5

$L_b :$	1	\mapsto	b	1	\mapsto	4
	a	\mapsto	a^2b	2	\mapsto	6
	a^2	\mapsto	ab	3	\mapsto	5
	b	\mapsto	1	4	\mapsto	1
	ab	\mapsto	a^2	5	\mapsto	3
	a^2b	\mapsto	a	6	\mapsto	2
$L_{ab} :$	1	\mapsto	ab	1	\mapsto	5
	a	\mapsto	b	2	\mapsto	4
	a^2	\mapsto	a^2b	3	\mapsto	6
	b	\mapsto	a	4	\mapsto	2
	ab	\mapsto	1	5	\mapsto	1
	a^2b	\mapsto	a^2	6	\mapsto	3
$L_{a^2b} :$	1	\mapsto	a^2b	1	\mapsto	6
	a	\mapsto	ab	2	\mapsto	5
	a^2	\mapsto	b	3	\mapsto	4
	b	\mapsto	a^2	4	\mapsto	3
	ab	\mapsto	a	5	\mapsto	2
	a^2b	\mapsto	1	6	\mapsto	1

Therefore we have the following map $\psi : D_3 \rightarrow S_6$:

$$\begin{array}{lll} 1 & \mapsto & (1) \\ b & \mapsto & (14)(26)(35) \\ a & \mapsto & (123)(456) \\ ab & \mapsto & (15)(24)(36) \\ a^2 & \mapsto & (132)(465) \\ a^2b & \mapsto & (16)(25)(34) \end{array}$$

By Cayley's theorem $D_3 \cong \{(1), (123)(456), (132)(465), (14)(26)(35), (15)(24)(36), (16)(25)(34)\}$.

- 4. (20 points)** Let $H = \{1, x^2\}$ (H is a subgroup of D_4).

- (a) Find all of the left and right cosets of H in D_4 . Is H a normal subgroup of D_4 ?

$|D_4| = 8$ and $|H| = 2$ so $[D_4 : H] = 8/2 = 4$ (there are 4 cosets of H in D_4). $x \notin H$ so xH will be a new coset. $xH = \{x, x^3\}$. Next, pick y , so yH will be a new coset $yH = \{y, yx^2 = x^2y\}$. Only xy and x^3y remain, so they must form our fourth and final left coset. $xyH = \{xy, x^3y\}$. What about right cosets?

- $H = \{1, x^2\}$ (A subgroup itself is always both a left and right coset.)
- $Hx = \{x, x^3\} = xH$
- $Hy = \{y, x^2y\} = yH$
- $Hxy = \{xy, x^3y\} = xyH$

Since the left and right cosets match, $H \triangleleft D_4$ (actually $H = Z(D_4)$ and the center of a group is always a normal subgroup).

(b) Write out a Cayley table for the quotient group D_4/H .

We know the coset H is the identity. Let's compute the other products (keep in mind that as we compute these products we want to end up with a representative from our list H, xH, yH, xyH).

- $xHxH = xxH = x^2H = H$ (since $x^2 \in H, x^2H = H$)
- $xHyH = xyH$
- $xHxyH = x^2yH = yH$ (since $x^2y \in yH, x^2yH = yH$)
- $yHxH = yxH = x^3yH = xyH$ (since $x^3y \in xyH, x^3yH = xyH$)
- $yHxyH = yxyH = x^3H = xH$ (since $x^3 \in xH, x^3H = xH$)
- $xyHxH = xyxH = yH$
- $xyHyH = xyyH = xH$
- $xyHxyH = 1H = H$ (since $1 \in H, 1H = H$)

Of course, we didn't really need to compute *all* of these products. After computing a few, we could use the "Latin square" properties (i.e. each row and column lists each element exactly once) of the Cayley table to finish filling out our rows and columns.

D_4/H	H	xH	yH	xyH
H	H	xH	yH	xyH
xH	xH	H	xyH	yH
yH	yH	xyH	H	xH
xyH	xyH	yH	xH	H

Notice that this is an Abelian group of order 4 which is not cyclic ($(xH)^2 = (yH)^2 = (xyH)^2 = H$ so the non-identity elements have order 2). D_4/H (and any group isomorphic to it) is called the Klein four-group.

5. (20 points) Let $\varphi : G \rightarrow G'$ be a homomorphism.

(a) Prove that $\text{Ker}(\varphi)$ is a subgroup of G . Then show that $\text{Ker}(\varphi)$ is a **normal** subgroup of G .

Let e be the identity of G and e' be the identity of G' . We know that $\varphi(e) = e'$ (homomorphisms map identities to identities). Therefore, $e \in \text{Ker}(\varphi)$ and thus $\text{Ker}(\varphi) \neq \emptyset$.

Let $x, y \in \text{Ker}(\varphi)$. Thus $\varphi(x) = e'$ and $\varphi(y) = e'$. Notice that $\varphi(xy) = \varphi(x)\varphi(y) = e'e' = e'$. Therefore, $xy \in \text{Ker}(\varphi)$ (closure).

Let $x \in \text{Ker}(\varphi)$. Thus $\varphi(x) = e'$ so $\varphi(x^{-1}) = (\varphi(x))^{-1} = (e')^{-1} = e'$. Therefore, $x^{-1} \in \text{Ker}(\varphi)$ (inverses).

Therefore, $\text{Ker}(\varphi)$ is a subgroup of G . Let's show that it is normal in G . Let $g \in G$ and $x \in \text{Ker}(\varphi)$. Thus $\varphi(x) = e'$. Consider, $\varphi(gxg^{-1}) = \varphi(g)\varphi(x)\varphi(g^{-1}) = \varphi(g)e'(\varphi(g))^{-1} = \varphi(g)(\varphi(g))^{-1} = e'$. Therefore, $gxg^{-1} \in \text{Ker}(\varphi)$ and so $\text{Ker}(\varphi) \triangleleft G$.

(b) Show that $\varphi(G)$ is a subgroup of G' .

Again let e and e' be the identities of G and G' respectively. First, note that $e' = \varphi(e) \in \varphi(G)$. Thus $\varphi(G)$ is a non-empty subset of G' .

Now let $y_1, y_2 \in \varphi(G)$. This implies there exists $x_1, x_2 \in G$ such that $\varphi(x_1) = y_1$ and $\varphi(x_2) = y_2$. Notice that $y_1y_2 = \varphi(x_1)\varphi(x_2) = \varphi(x_1x_2) \in \varphi(G)$ since $x_1x_2 \in G$. Therefore, $\varphi(G)$ is closed under the operation.

Finally, let $y \in \varphi(G)$ so there exists some $x \in G$ such that $\varphi(x) = y$. Then $y^{-1} = (\varphi(x))^{-1} = \varphi(x^{-1}) \in \varphi(G)$ since $x^{-1} \in G$. Therefore, $\varphi(G)$ is closed under inversion.

Therefore, $\varphi(G)$ is a subgroup of G' .

D_3	1	a	a^2	b	ab	a^2b
1	1	a	a^2	b	ab	a^2b
a	a	a^2	1	ab	a^2b	b
a^2	a^2	1	a	a^2b	b	ab
b	b	a^2b	ab	1	a^2	a
ab	ab	b	a^2b	a	1	a^2
a^2b	a^2b	ab	b	a^2	a	1

D_4	1	x	x^2	x^3	y	xy	x^2y	x^3y
1	1	x	x^2	x^3	y	xy	x^2y	x^3y
x	x	x^2	x^3	1	xy	x^2y	x^3y	y
x^2	x^2	x^3	1	x	x^2y	x^3y	y	xy
x^3	x^3	1	x	x^2	x^3y	y	xy	x^2y
y	y	x^3y	x^2y	xy	1	x^3	x^2	x
xy	xy	y	x^3y	x^2y	x	1	x^3	x^2
x^2y	x^2y	xy	y	x^3y	x^2	x	1	x^3
x^3y	x^3y	x^2y	xy	y	x^3	x^2	x	1