**Math 3110** **Test #2** **March $31^{st}$, 2010**

**Name:** ________________________________ **Be sure to show your work!**

**1. (____/20 points)** Random Group Stuff — Fill out the following table:

| $G =$ | What is the identity of $G$? | Is $G$ abelian? | Is $G$ cyclic? | What is the order of ...? | Does $G$ have an element of order 4? |
|---|---|---|---|---|---|
| $\mathbb{Z}_{40}$ | | | | $\|30\| =$ | |
| $U(8)$ | | | | $\|5\| =$ | |
| $D_7$ | | | | $\|x^4y\| =$ | |
| $S_4$ | | | | $\|(12)(34)\| =$ | |

**Recall:** $D_7 = \{1, x, \ldots, x^6, y, xy, \ldots, x^6y\}$ where $x^7 = 1$, $y^2 = 1$, and $xyxy = 1$.

Scratch Work:

**2. (____/20 points)** Group or not? Are the following sets with operations groups or not? If $G$ is a group, prove it — you may use a subgroup test if it applies. If $G$ fails to be a group, explain what property fails.

(a) Let $G = [-1, 1] = \{r \in \mathbb{R} \mid -1 \leq r \leq 1\}$ with the operation "+" (addition).

(b) Let $G = \left\{ \begin{bmatrix} 1 & 0 \\ r & 1 \end{bmatrix} \,\middle|\, r \in \mathbb{R} \right\}$ with the operation of matrix multiplication.

**3. (____/15 points)** Cayley's Theorem and Permutations.

Recall that $D_n = \{1, x, x^2, \ldots, x^{n-1}, y, xy, \ldots, x^{n-1}y\}$ where $x^n = 1$, $y^2 = 1$, and $xyxy = 1$.

(a) Write down what the left multiplication operator of $y$ does in $D_3$. Then write down the corresponding permutation if we label 1 as 1, $x$ as 2, $x^2$ as 3, $y$ as 4, $xy$ as 5, $x^2y$ as 6.

$L_y : D_3 \to D_3$

$$\begin{aligned} 1 &\mapsto \underline{\hspace{3cm}} \\ x &\mapsto \underline{\hspace{3cm}} \\ x^2 &\mapsto \underline{\hspace{3cm}} \\ y &\mapsto \underline{\hspace{3cm}} \\ xy &\mapsto \underline{\hspace{3cm}} \\ x^2y &\mapsto \underline{\hspace{3cm}} \end{aligned}$$

The corresponding permutation is...?

(b) Suppose that using Cayley's theorem we found the left multiplication operator of $x$ in $D_4$ corresponds to $(1234)(5678)$ and $y$ corresponds to $(15)(28)(37)(46)$. What would the left multiplication operator of $x^2y$ correspond to? [Your answer should be a permutation written as a product of disjoint cycles.]

Now write your answer as a product of transpositions. Is this permutation even or odd?

**4. (____/25 points)** Mod stuff.

(a) Draw the subgroup lattice of $\mathbb{Z}_{44}$. *Note:* $44 = 2^2 \cdot 11$.

(b) List the possible orders of elements in $\mathbb{Z}_{44}$. Then determine the number of elements of each order.

| Order = | | | | | | |
|---|---|---|---|---|---|---|
| Number of elements = | | | | | | |

(c) Show that $f : \mathbb{Z}_6 \to \mathbb{Z}_4$ defined by $f(x) = 2x$ is a **well-defined** homomorphism.

(d) $\text{Ker}(f) =$ ________________________________

$f(\mathbb{Z}_6) =$ ________________________________

Is $f$ 1-1?

Is $f$ onto?

Is $f$ an isomorphism?

**5. (____/20 points)** POOF! ...I mean... PROOFS! [No magic please.]

(a) Let $G$ be a group and let $a, b \in G$ such that $(ab)^2 = a^2b^2$. Show that $ab = ba$.

(b) Let $G$ be a group and let $g \in G$. Define the map $\varphi : G \to G$ by $\varphi(x) = gxg^{-1}$. Prove that $\varphi$ is an isomorphism.

(c) Let $G$ and $G'$ be groups and let $\psi : G \to G'$ be a homomorphism. Suppose that $G$ is cyclic. Show that $\psi(G)$ is abelian. [Extra Credit: Show that $\psi(G)$ is cyclic.]