Math 3110 # Test #3 **April 20^{th}, 2010**

Name: ______________________________ **Be sure to show your work!**

1. (____/20 points) For each of the following pairs of groups, if the groups are isomorphic, circle $G_1 \cong G_2$ and explain why they are isomorphic. If the groups aren't isomorphic, circle $G_1 \ncong G_2$ and explain why not.

(a) $\mathbb{C} \cong \mathrm{GL}_2(\mathbb{R})$ OR $\mathbb{C} \ncong \mathrm{GL}_2(\mathbb{R})$

(b) $\mathbb{Z}_6 \cong U(7)$ OR $\mathbb{Z}_6 \ncong U(7)$

(c) $S_4 \cong D_{12}$ OR $S_4 \ncong D_{12}$

2. (____/20 points) A very normal problem. [Assume G is a group with identity e.]

(a) Let H and K be normal subgroups of G. Show that $H \cap K$ is a normal subgroup of G.

(b) Let H be a subgroup of G and let N be a normal subgroup of G.
Let $HN = \{hn \,|\, h \in H \text{ and } n \in N\}$ The following facts are true (you do not have to prove them): HN is a subgroup of G and N is a normal subgroup of HN.
Let $\phi : H \to HN/N$ be defined by $\phi(h) = hN$.
This map is onto since $hnN \in HN/N$ implies that $\phi(h) = hN = hnN$ ($n \in N$ implies $nN = N$).

i. Show ϕ is a homomorphism.

ii. Show $\text{Ker}(\phi) = H \cap N$.

iii. State the first isomorphism theorem. Then apply this theorem to ϕ.

3. (____/20 points) Computing with Quotients.

(a) Recall $A_4 = \{(1), (123), (132), (124), (142), (134), (143), (234), (243), (12)(34), (13)(24), (14)(23)\}$. Let $H = \{(1), (12)(34), (13)(24), (14)(23)\}$. H is a subgroup of A_4 (you don't need to prove this). Find all of the left and right cosets of H. Is H normal in A_4? If so, write a Cayley table for A_4 / H and determine if this quotient is abelian, cyclic, both or neither. If H is not normal in A_4, explain why not.

(b) Compute the index of $\langle 4 \rangle$ in $\mathbb{Z}_{12}$. Then write down all of the cosets of $\langle 4 \rangle$ in $\mathbb{Z}_{12}$. Finally, write a Cayley table for $\mathbb{Z}_{12} / \langle 4 \rangle$. Is this quotient is abelian, cyclic, both or neither?

4. (____/20 points) Consider the ring $\mathbb{Z}_{10}$. For each of the following elements circle "unit" if it's a unit and/or "zero divisor" if it's a zero divisor. If an element has a (multiplicative) inverse, find it. If it's a zero divisor, show why it's a zero divisor.

1 is a unit zero divisor

2 is a unit zero divisor

3 is a unit zero divisor

4 is a unit zero divisor

5 is a unit zero divisor

6 is a unit zero divisor

7 is a unit zero divisor

8 is a unit zero divisor

9 is a unit zero divisor

A few quick questions...

(a) Is $\mathbb{Z}_{10}$ a commutative ring with unity? (Just "Yes" or "No" will suffice.)

(b) Is $\mathbb{Z}_{10}$ an integral domain? Why? or Why not?

(c) Is $\mathbb{Z}_{10}$ a field? Why or Why not?

5. (____/20 points) Some quick ring questions.

(a) Briefly explain (in a sentence or two) why all fields are integral domains.

(b) Let $\mathbb{F}$ be a field, let $r \in \mathbb{F}$, and suppose $r^2 = r$. Prove that $r = 0$ or $r = 1$.
Is this still true if $\mathbb{F}$ is an integral domain?

(c) Show $S = \left\{ \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} \;\middle|\; x \in \mathbb{R} \right\}$ is a subring of $\mathbb{R}^{2\times 2}$ (the ring of 2×2 real matrices).