Math 3110 # Test #1 **February 11^{th}, 2015**

Name: ____________________________________ **Be sure to show your work!**

1. (20 points) Definition and Basics

(a) Suppose that G is a non-empty set equipped an operation. What 4 things do I need to check to see if G is a group? Give details.

1:

2:

3:

4:

What additional property needs to hold for G to be an **Abelian** group?

5:

(b) Let $G = \mathbb{Z}_{\geq 0}$ be the set of non-negative integers. It can be shown that $x \star y = \max\{x, y\}$ (example: $3 \star 1 = \max\{3, 1\} = 3$) is an associative, commutative (closed) binary operation on G with identity 0. However, G is not a group. Why? [Use a concrete counterexample.]

(c) Let $G = \mathbb{R} - \mathbb{Q} = \{x \in \mathbb{R} \mid x \text{ is irrational } \}$. Prove G is **not** a group under addition.

2. (20 points) Some modular arithmetic.

(a) What is the inverse of 10 in the group $\mathbb{Z}_{15}$? What operation makes $\mathbb{Z}_{15}$ a group?

(b) Is 10 an element of $U(15)$? If not, why not? If so, why so & what is its inverse?

(c) List all of the *distinct* cyclic subgroups, $\langle x \rangle$, of $\mathbb{Z}_{15}$.

(d) What is the order of 10 in $\mathbb{Z}_{15}$?

(e) Draw the subgroup lattice of $\mathbb{Z}_{15}$.

3. (25 points) More Modular Arithmetic.

(a) Draw a Cayley table for $U(5)$.

(b) List the elements of $U(5)$. Then find their **order**s and the list the elements in **cyclic subgroup** generated by that element. [*Note:* There may be more spaces than you need.]

$x =$						
$\lvert x\rvert =$						
$\langle x\rangle =$	$\{\quad\}$	$\{\quad\}$	$\{\quad\}$	$\{\quad\}$	$\{\quad\}$	$\{\quad\}$

(c) Find $\langle A\rangle = \left\langle \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \right\rangle$ in $\mathrm{GL}_2(\mathbb{Z}_6)$. What is the order of A? What is A^{-1}?

(d) ~~Find 99^{-1} mod 123 using~~ Run the extended Euclidean algorithm on 99 and 123.

4. (15 points) Recall $D_n = \{1, x, \ldots, x^{n-1}, y, xy, \ldots, x^{n-1}y\} = \langle x, y \mid x^n = 1, y^2 = 1, (xy)^2 = 1\rangle$.

(a) Use the relations for D_{10} to simplify $x^{-3}y^2x^{15}yx^4y^{-22}$

(b) Fill in the Cayley table for D_3:

	1	x	x^2	y	xy	x^2y
1	1	x	x^2	y	xy	x^2y
x	x					
x^2	x^2					
y	y					
xy	xy					
x^2y	x^2y					

(c) Do the **reflections** form a subgroup of D_3? If so, why? If not, why not?

5. (20 points) Proofs!

(a) Choose one of the following:

I. Suppose that $(ab)^{-1} = a^{-1}b^{-1}$ for all $a, b \in G$. Prove that G is abelian.

II. Let G be a group and $g \in G$. Prove that $\langle g \rangle = \langle g^{-1} \rangle$ (i.e. g and its inverse generate the same cyclic subgroup).

(b) Choose one of the following: (You **must** use a subgroup test in your proof.)

I. Prove that $H = 6\mathbb{Z} = \{m \in \mathbb{Z} \mid m \text{ is a multiple of } 6\ \}$ is a subgroup of $\mathbb{Z}$.

II. Prove that $K = \left\{ \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \;\middle|\; a \in \mathbb{R} \right\}$ is a subgroup of $\mathrm{SL}_2(\mathbb{R})$.