Math 3110 **Test #3** **April 10^{th}, 2015**

Name: ________________________________ **Be sure to show your work!**

1. (15 points) Getting things in order...

(a) Let $G = D_3 \oplus Q$ where $D_3 = \langle x, y \mid x^3 = 1, y^2 = 1, xyxy = 1 \rangle = \{1, x, x^2, y, xy, x^2y\}$ is a dihedral group and $Q = \{\pm 1, \pm i, \pm j, \pm k\}$ is the group of quaternions.

The order of G is $|G| =$ ____________

Does G have an element of order 12? If so, give an example. If not, explain why not.

(b) Let G be a group with subgroups H and K such that $H \subseteq K \subseteq G$. In addition, suppose that $|H| = 2$ and $|G| = 30$. What are the possible orders of K?

2. (10 points) Let $\varphi : G \to H$ be a homomorphism between two groups G and H. Prove that $\ker(\varphi) \triangleleft G$.

[You may **not** assume that the kernel is a subgroup – prove this as well.]

3. (25 points) Quotients

(a) Given: $H = \{(1), (12)(34), (13)(24), (14)(23)\}$ is a normal subgroup of S_4.

The order of ${S_4}/{H}$ is ____________________________.

The identity of ${S_4}/{H}$ is ______________. $((1234)H)^{-1} =$ ____________________________.

The order of $(1234)H$ in ${S_4}/{H}$ is ______________. The size of the set $(1234)H$ is ______________.

Scratch work:

(b) Consider ${\mathbb{Z}_{20}}/{H}$ where $H = \langle 4 \rangle = \{0, 4, 8, 12, 16\}$. List all of the cosets (and their contents) of H in $\mathbb{Z}_{20}$. Then make a Cayley table for this quotient group.

4. (15 points) Let $\varphi : \mathbb{Z}_6 \to \mathbb{Z}_{10}$ be defined by $\varphi(x) = 5x$.

(a) Show that φ is a homomorphism. [Do we need to prove that φ is well-defined?]

(b) Compute the kernel and image of φ.

(c) When applied to this φ, what does the first isomorphism theorem tell us?

5. (10 points) Let $H, K \triangleleft G$ and $G = HK$. The second isomorphism theorem says that $G/K \cong H/H \cap K$. We proved this in class using the function $\varphi : H \to G/K$ defined by $\varphi(x) = xK$.

(a) What needs to be proven about φ to establish this theorem?

(b) Determine the kernel of φ.

6. (25 points) Finite Abelian Groups

(a) List all of the non-isomorphic abelian groups of order $36 = 2^2 3^2$. Circle any that are cyclic.

(b) How many non-isomorphic abelian groups of order 25,508,082,600 are there?

Note: $25{,}508{,}082{,}600 = 2^3 3^4 5^2 7^1 11^3 13^2$ and there are 5 non-isomorphic abelian groups of order $3^4 = 81$. ☺

(c) Are the groups $\mathbb{Z}_6 \oplus \mathbb{Z}_{20} \oplus \mathbb{Z}_9$ and $\mathbb{Z}_{30} \oplus \mathbb{Z}_{36}$ isomorphic? Explain your answer.

(d) Is the group $\mathbb{Z}_2 \oplus \mathbb{Z}_{21} \oplus \mathbb{Z}_{15}$ cyclic? Explain you answer.