Math 3110 **Test #1** **February 17^{th}, 2021**

Name: ______________________________ **Be sure to show your work!**

1. (20 points) Definition and Basics

(a) Suppose that G is a non-empty set equipped an operation. What 4 things do I need to check to see if G is a group? Give details.

1:

2:

3:

4:

What additional property needs to hold for G to be an **Abelian** group?

5:

(b) The positive real numbers $\mathbb{R}_{>0} = \{r \in \mathbb{R} \mid r > 0\}$ do not form a group under division. Why not?

(c) On the other hand, the positive real numbers $\mathbb{R}_{>0} = \{r \in \mathbb{R} \mid r > 0\}$ do form a group if we select the right operation. Which operation turns this collection of numbers into a group: Addition or Multiplication? Then explain why the other operation does not yield a group.

(d) The non-zero rational numbers $\mathbb{Q}_{\neq 0}$ form a group under multiplication. On the other hand, the (non-zero) irrational numbers $\mathbb{I} = \mathbb{R} - \mathbb{Q} = \{x \in \mathbb{R} \mid x \notin \mathbb{Q}\}$ do not. Why?

2. (20 points) Some modular arithmetic.

(a) Make a list of all of the cyclic subgroups of $\mathbb{Z}_{10}$ along with their contents (for example: $\langle 0 \rangle = \{0\}$).

(b) Fill out the following table referring to the operations of addition and multiplication modulo 10:
Note: Just put an **X** if something is undefined / does not exist.

Element $x =$	0	1	2	3	4	5	6	7	8	9
Additive Inverse $-x =$										
Order (in $\mathbb{Z}_{10}$) $\lvert x \rvert =$										
Multiplicative Inverse $x^{-1} =$										
Order (in $U(10)$) $\lvert x \rvert =$										

(c) Compute $2^{-1}(3-8)+7 \bmod 10$ or explain why this is undefined.

(d) Compute $3^{-1}(7-3)-11 \bmod 10$ or explain why this is undefined.

3. (20 points) More Modular Arithmetic.

(a) Write down a Cayley table for $U(8)$. Is $U(8)$ cyclic (circle the correct answer)? Yes / No

(b) Draw the subgroup lattice for $\mathbb{Z}_{20}$. $[20 = 2^2 \cdot 5]$

(c) Find $10^{-1} \bmod 77$ using the extended Euclidean algorithm [Don't just guess and check].

4. (20 points) Recall $D_n = \{1, x, \ldots, x^{n-1}, y, xy, \ldots, x^{n-1}y\} = \langle x, y \mid x^n = 1, y^2 = 1, (xy)^2 = 1 \rangle$.

(a) Use the relations for D_8 to simplify $x^{13}y^3x^{-2}y^{888}x$

(b) Make a table listing the elements of D_8, their inverses, and their orders.

(c) What is $\langle x^6 \rangle$ in D_8?

(d) Fill in the following rows of the Cayley table for D_4:

	1	x	x^2	x^3	y	xy	x^2y	x^3y
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
x^3								
y								
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

(e) Is $H = \{1, y, x^2y\}$ a subgroup of D_4? Why or why not?

5. (20 points) Proofs!

(a) Choose one of the following: Assume G is a group under multiplication with identity 1.

I. Suppose that $g = g^{-1}$ for all $g \in G$. Prove that G is abelian.

II. Suppose that $G = \langle g \rangle$ is a cyclic group. Prove that G is abelian.

(b) Choose one of the following: (You **must** use a subgroup test in your proof.)

I. Prove that $H = 8\mathbb{Z} = \{8k \mid k \in \mathbb{Z}\}$ is a subgroup of $\mathbb{Z}$.

II. Prove that $K = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \,\middle|\, a, b \in \mathbb{R}_{\neq 0} \right\}$ is a subgroup of $\mathrm{GL}_2(\mathbb{R})$.