

Name: _____

Be sure to show your work!

1. (15 points) Getting things in order...

- (a) Let $G = Q \times A_4$ where $Q = \{\pm 1, \pm i, \pm j, \pm k\}$ is the quaternion group and A_4 is the group of even permutations in S_4 (i.e., $A_4 = \{(1), (123), (132), (124), (142), (134), (143), (234), (243), (12)(34), (13)(24), (14)(23)\}$).

The order of G is $|G| =$ _____What is the largest element order in $Q \times A_4$? **Give an example** of such an element.

- (b) Let G be a group of order 50 with subgroups H , K , and L . In addition, suppose the order of H is 5, the order of L is 2, and that $H \subseteq K \subseteq G$.

What is/are the possible order(s) of K ?What can we conclude about $H \cap L$?2. (15 points) Let $\varphi : G \rightarrow H$ be a homomorphism.

→ State the definition a homomorphism.

→ State the definition of the kernel: $\ker(\varphi)$.→ Then prove that $\ker(\varphi)$ is a normal subgroup of G . *Note:* Prove it is a subgroup and that it is normal in G .

3. (15 points) Consider $K = \{1, x^2, x^4\}$ in $D_6 = \langle x, y \mid x^6 = 1, y^2 = 1, xyxy = 1 \rangle = \{1, x, \dots, x^5, y, xy, \dots, x^5y\}$.
Note: It can be shown that K is a normal subgroup of D_6 (just accept this for now).

(a) Quick questions about D_6/K .

The order of D_6/K is _____. List the distinct elements of $D_6/K = \left\{ \frac{\quad}{K} \right\}$.
 [In terms of K , for example: " xK ".]

The identity of D_6/K is _____. $(xK)^{-1} = \frac{\quad}{K}$.
 [Simplify please.]

The order of xK in D_6/K is _____. List the contents of $xK = \left\{ \frac{\quad}{K} \right\}$.

Scratch work:

- (b) Let $H = \{1, x^3, y, x^3y\}$. While H is a subgroup of D_6 , it is **not** a normal subgroup of D_6 . Prove it isn't normal.

4. (10 points) Consider \mathbb{Z}_{12}/H where $H = \langle 3 \rangle = \{0, 3, 6, 9\}$.
 → List all of the cosets of H (and their contents) in \mathbb{Z}_{12} .
 → Then make a Cayley table for this quotient group.

What is the order of $8 + H$ in \mathbb{Z}_{12}/H ?

5. (15 points) Something is terribly, horribly wrong!

(a) Let $H \triangleleft \mathbb{Z}_{100}$. Why is $\frac{\mathbb{Z}_{100}}{H} \cong \mathbb{Z}_5 \times \mathbb{Z}_5$ impossible?

(b) Let $\varphi : A_4 \rightarrow Q$ be a homomorphism (where A_4 and Q are the same as in problem 1(a)). Why can't φ be onto?

(c) Recall $H = \{1, x^3, y, x^3y\}$ (a subgroup of D_6) from problem 3(b). Suppose $\varphi : D_6 \rightarrow \mathbb{Z}$ is a homomorphism. Why can't $\ker(\varphi) = H$?

6 (10 points) Let H be a subgroup of G where G is an Abelian group. First, prove that H is in fact a normal subgroup. Then, prove that $\frac{G}{H}$ is Abelian.

7. (20 points) Finite Abelian Groups

(a) List all of the non-isomorphic Abelian groups of order $100 = 2^2 5^2$. Circle any that are cyclic.

(b) How many non-isomorphic Abelian groups of order 449,878,000 are there?

Note: $449,878,000 = 2^4 \cdot 5^3 \cdot 11^3 \cdot 13^2$ and there are 5 non-isomorphic Abelian groups of order $16 = 2^4$. ☹

(c) Are the groups $\mathbb{Z}_6 \times \mathbb{Z}_{10} \times \mathbb{Z}_{10}$ and $\mathbb{Z}_{12} \times \mathbb{Z}_{50}$ isomorphic? Explain your answer.

(d) What is the largest order among elements of $\mathbb{Z}_6 \times \mathbb{Z}_9 \times \mathbb{Z}_{15}$? Explain your answer.