**Math 3110** **Final Exam** **May $2^{\text{nd}}$, 2022**

**Name:** ______________________________ **Be sure to show your work!**

1. (15 points) Working in $\mathbb{Z}_{12}$.

(a) $I = (4) = \langle 4 \rangle = \Big\{$ ________________________ $\Big\}$ and $\mathbb{Z}_{12}/I = \Big\{$ ________________________ $\Big\}$.

(b) Write addition and multiplication tables for $\mathbb{Z}_{12}/I$.

(c) For each element in $\mathbb{Z}_{12}/I$, state whether that element is zero, a zero divisor, a unit, or none of the above. If it is a unit, give its inverse. If it is an zero divisor, show that this is the case.

(d) Suppose $R$ is some ring and $\varphi : \mathbb{Z}_{12} \to R$ is a homomorphism. Let $K = \mathrm{Ker}(\varphi)$. Give a complete list of all possible $K$'s. Circle the choice(s) that guarantee $\varphi$ is one-to-one.

(e) Which ideals $I$ in $\mathbb{Z}_{12}$ make sure $\mathbb{Z}_{12}/I$ is a field?

2. (12 points) Groups: Explain why each pair of **groups** are not isomorphic.

(a) Explain why $A_5 \not\cong D_{30}$ [not isomorphic]. *Note:* $A_5$ is the group of even permutations in $S_5$.

(b) Explain why $S_4 \not\cong \mathbb{Z}_2 \times \mathbb{Z}_{12}$ [not isomorphic].

(c) Explain why $U(5) \not\cong U(8)$ [not isomorphic].

3. (8 points) Rings: Explain why each pair of **rings** are not isomorphic.

(a) $\mathbb{R}[x] \not\cong \mathbb{R}^{2\times 2}$ [not isomorphic]. *Note:* These are real polynomials vs. $2 \times 2$ real matrices.

(b) $\mathbb{Q} \not\cong \mathbb{Z}$ [not isomorphic].

4. (8 points) Workin' in $\mathbb{Z}_{36}$. [Note: $36 = 2^2 \cdot 3^2$]

(a) Fill out the following table (for $\mathbb{Z}_{36}$):

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| order = | | | | | | | | | |
| number of elements with this order = | | | | | | | | | |

(b) Draw $\mathbb{Z}_{36}$'s lattice of ideals.

(c) Which ideals are prime? maximal?

5. (6 points) Workin' in $\mathbb{Z}_{110}$.

(a) Is 88 zero, a unit, a zero divisor, or none of the above in $\mathbb{Z}_{110}$? If 88 is a zero divisor, prove it. If 88 is a unit, find its inverse. If none of the above, explain why so.

(b) Is 53 zero, a unit, a zero divisor, or none of the above in $\mathbb{Z}_{110}$? If 53 is a zero divisor, prove it. If 53 is a unit, find its inverse. If none of the above, explain why so.

6. (17 points) Sub-things

(a) Recall that $D_8 = \{1, x, \ldots, x^7, y, xy, \ldots, x^7y\} = \langle x, y \mid x^8 = 1, y^2 = 1, xyxy = 1\rangle$.
Let $H = \{1, x^4, y, x^4y\}$. Explain why $H$ is a subgroup but **not** a **normal** subgroup of $D_8$.

(b) Let $G$ be an Abelian group with identity 1. Prove that $H = \{g \in G \mid g^2 = 1\}$ is a subgroup of $G$.

(c) Let $\mathbb{Q}[i] = \{a + bi \mid a, b \in \mathbb{Q}\}$ where $i^2 = -1$. Show that $\mathbb{Q}[i]$ is a sub**field** of the complex numbers $\mathbb{C}$.

(d) Let $R$ be a finite commutative ring with 1 and let $I$ be an ideal of $R$. Is it possible for $I$ to be prime and not maximal? Explain your answer.

7. (15 points) An ideal question.

(a) Recall that $\mathbb{Z}[x]$ is the ring of polynomials with integer coefficients. Let $I = \{f(x) \in \mathbb{Z}[x] \mid f(0) \text{ is even}\}$. Prove that $I$ is an ideal of $\mathbb{Z}[x]$.

(b) Let $I$ and $J$ be ideals of some ring $R$. Prove that $I \cap J$ is an ideal of $R$.

(c) Suppose $R$ is some ring and $\varphi : \mathbb{Z}_{10} \to R$ is an **onto** ring homomorphism. What can we say about the size of $R$? What can we say about $R$ being commutative or not?

8. (7 points) The Fundamental Theorem of Finite Abelian Groups.

(a) List all of the non-isomorphic abelian groups of order $72 = 2^3 \cdot 3^2$. Circle any that are cyclic.

(b) Which of the abelian groups of order 72 contain elements of order 36?

9. (6 points) Suppose $G$ is a group of order 60 with subgroups $H$ and $K$. Moreover, suppose the order of $H$ is 15 and that $H \subsetneq K \subsetneq G$. Explain why $K$ must be a normal subgroup and why $G/K$ must be cyclic.

10. (6 points) Let $\varphi : \mathbb{Q}[x] \to \mathbb{C}$ be defined by $\varphi(f(x)) = f(i)$ where $i^2 = -1$. **(a): Prove** $\varphi$ is a ring homomorphism. Note that the range of $\varphi$ is $\mathbb{Q}[i]$ and $\text{Ker}(\varphi) = (x^2 + 1)$ (Bonus: prove this). **(b): What** does the first isomorphism theorem say here? Bonus: Is $(x^2 + 1)$ a prime and/or maximal ideal of $\mathbb{Q}[x]$?