Math 3110 **Test #3** **April 4^{th}, 2022**

Name: Answer Key **Be sure to show your work!**

1. (15 points) Getting things in order...

(a) Let $G = \mathbb{Z}_{10} \times S_3 \times Q$ where S_3 is the group of permutations on 3 elements and $Q = \{\pm 1, \pm i, \pm j, \pm k\}$ is the quaternion group.

The order of G is $|G| = |\mathbb{Z}_{10}| \cdot |S_3| \cdot |Q| = 10 \cdot 6 \cdot 8 = \boxed{480}$

What is the largest element order in $\mathbb{Z}_{10} \times S_3 \times Q$? **Give an example** of such an element.

Elements in $\mathbb{Z}_{10}$ have orders 1, 2, 5, and 10. Elements in S_3 have orders 1, 2, and 3 (the identity, transpositions, and 3-cycles). Elements in Q have orders 1, 2, and 4. The orders of elements in G are then least common multiples of such numbers. Using 5, 3, and 4 we can achieve order 60 (using 10, 3, and 4 does this as well). Therefore, the largest order appearing among elements of G is $\text{lcm}(5, 3, 4) = 60$. For example, $\boxed{(2, (123), i) \text{ has order } 60}$. Of course, there are many many other elements with this order, for example $|(1, (132), -k)| = \text{lcm}(10, 3, 4) = 60$ works too!

(b) Let H and K be subgroups of a group G. In addition suppose that $|H| = 15$ and $|K| = 8$. Explain why $H \cap K$ must be the trivial subgroup.

We know that $H \cap K$ is a subgroup of both H and K. By Lagrange's Theorem, its order must divide both $|H| = 15$ and $|K| = 8$. Since 15 and 8 are relatively prime, $|H \cap K| = 1$ is the only possibility. Thus $H \cap K$ is the trivial subgroup.

2. (15 points) Let $\varphi : G \to H$ be a homomorphism, and let G, H, and K denote groups.

$\longrightarrow$ State the definition a homomorphism.

We say φ is a homomorphism if $\varphi(ab) = \varphi(a)\varphi(b)$ for all $a, b \in G$.

$\longrightarrow$ State the definition of the kernel: $\ker(\varphi)$.

The kernel is the set of everything in the domain that maps to the identity of the codomain: $\ker(\varphi) = \{x \in G \mid \varphi(x) = e\}$.

$\longrightarrow$ Then prove $\pi : K \times H \to H$ defined by $\pi(k, h) = h$ is a homomorphism and determine its kernel (=??×??).

Let $(a, b), (x, y) \in K \times H$. Then $\pi((a, b)(x, y)) = \pi(ax, by) = by = \pi(a, b)\pi(x, y)$. Next,

$$\ker(\pi) = \{(x, y) \in K \times H \mid \pi(x, y) = e\} = \{(x, y) \in K \times H \mid y = e\} = \{(x, e) \mid x \in K\} = K \times \{e\}.$$

$\longrightarrow$ [Bonus question I meant to ask:] Notice π is onto. What does the first isomorphism theorem say?

We have $\ker(\pi) = K \times \{e\}$ and $\pi(K \times H) = H$ (since π is onto). Therefore, the first isomophism theorem (i.e., domain mod kernel is isomorphic to image) says: $(K \times H)/(K \times \{e\}) \cong H$.

3. (15 points) Consider $H = \{(1), (12)(34), (13)(24), (14)(23)\}$ in S_4 be the group of permutations on 4 elements. *Note:* It can be shown that H is a normal subgroup of S_4 (just accept this for now).

(a) Quick questions about S_4/H. The order of S_4/H is $\dfrac{|S_4|}{|H|} = \dfrac{24}{4} = \boxed{6}$.

$\boxed{\text{True}}$ / False: $(123)H = H(123)$ True / $\boxed{\text{False}}$: $(1234)H = (12)(34)H$

We know that $gH = Hg$ for all $g \in S_4$ since H is a *normal* subgroup of S_4 (i.e., its left and right cosets match). On the other hand, $(12)(34)H = H$ since $(12)(34) \in H$ whereas $(1234)H \neq H$ since $(1234) \notin H$. Thus these are not the same (left) coset.

The identity of S_4/H is H (or $(1)H$ works too). $(12)H = \{ (12), (34), (1324), (1423) \}$

Where we calculated: $(12)(1) = (12)$, $(12)(12)(34) = (34)$, $(12)(13)(24) = (1324)$, and $(12)(14)(23) = (1423)$.

The order of $(1234)H$ in S_4/H is 2. $((123)H)^{-1} = (123)^{-1}H = \boxed{(132)H}$.

The cardinality of $(1234)H$ is 4 – just like every coset of H in S_4. However, this coset's order in our quotient group is 2 since $(1234)H \neq H$ because $(1234) \notin H$, but $((1234)H)^2 = (1234)^2H = (13)(24)H = H$ because $(13)(24) \in H$.

(b) Let $H = \{1, x^2, y, x^2y\}$ and note that H is a subgroup of $D_4 = \langle x, y \mid x^4 = 1, y^2 = 1, (xy)^2 = 1\rangle = \{1, x, x^2, x^3, y, xy, x^2y, x^3y\}$. Is this a normal subgroup? Justify your answer.

$\boxed{\text{Yes}}$, $[D_4 : H] = \dfrac{|D_4|}{|H|} = \dfrac{8}{4} = 2$ and every subgroup of index 2 is normal. Alternatively, you could calculate its left and right cosets and see that they match: $H = \{1, x^2, y, x^2y\}$ is both a left and right coset. The other coset is $xH = \{x, x^3, xy, x^3y\} = Hx$ (i.e., the set complement of H in D_4).

4. (10 points) Consider $\mathbb{Z}_{24}/H$ where $H = \langle 4\rangle = \{0, 4, 8, 12, 16, 20\}$.

$\longrightarrow$ List all of the cosets of H (and their contents) in $\mathbb{Z}_{24}$.

Notice that there should be $|\mathbb{Z}_{24}|/|H| = 24/6 = 4$ cosets.

We have: $\boxed{H = \{0, 4, 8, 12, 16, 20\},\ 1 + H = \{1, 5, 9, 13, 17, 21\},\ 2 + H = \{2, 6, 10, 14, 18, 22\}, \text{ and } 3 + H = \{3, 7, 11, 15, 19, 23\}}$.

$\longrightarrow$ Then make a Cayley table for this quotient group.

	H	$1+H$	$2+H$	$3+H$
H	H	$1+H$	$2+H$	$3+H$
$1+H$	$1+H$	$2+H$	$3+H$	H
$2+H$	$2+H$	$3+H$	H	$1+H$
$3+H$	$3+H$	H	$1+H$	$2+H$

For example: $(2 + H) + (3 + H) = (2 + 3) + H = 5 + H = 1 + H$ since $5 \in 1 + H$.

What is the order of $3 + H$ in $\mathbb{Z}_{24}/H$?

We repeatedly add this coset to itself: $3 + H \neq H$ since $3 \in H$, $(3 + H) + (3 + H) = 6 + H = 2 + H \neq H$ since $2 \notin H$, $(3+H)+(3+H)+(3+H) = 9+H = 1+H \neq H$ since $1 \notin H$. Finally, $(3+H)+(3+H)+(3+H)+(3+H) = 12+H = H$ since $12 \in H$. Therefore, the order of $3 + H$ in this quotient group is $\boxed{4}$. Alternatively, once we see that $3 + H$ is not the identity (i.e., H itself) and $(3 + H) + (3 + H) \neq H$, we know its order isn't 1 or 2. The only remaining possibility (in our group of order 4) is 4 itself!

5. (15 points) Not quite proofs.

(a) Let $H \triangleleft \mathbb{Z}_{10} \times \mathbb{Z}_6$. Why is $\mathbb{Z}_{10} \times \mathbb{Z}_6/H \cong A_4$ impossible?

Looking purely at sizes, $|\mathbb{Z}_{10} \times \mathbb{Z}_6| = 10 \cdot 6 = 60$ and $|A_4| = 12$, so at first glance it seems that (given $|H| = 5$ so we'd have $60/5 = 12$) this is possible. However, $\boxed{\text{quotients of Abelian groups are Abelian}}$. Notice that $\mathbb{Z}_{60} \times \mathbb{Z}_6$ is Abelian but A_4 is not! *Note:* While it is true that quotients of cyclic groups are themselves cyclic, $\mathbb{Z}_{10} \times \mathbb{Z}_6$ is not cyclic! (This stems from the fact that 10 and 6 aren't relatively prime).

(b) Let $\varphi : D_4 \to \mathbb{Z}_{12}$ be a homomorphism. Why can't φ be one-to-one?

If φ was one-to-one, we would have the followings: $8 = |D_4| = |\ker(\varphi)| \cdot |\varphi(D_4)| = 1 \cdot |\varphi(D_4)|$. In other words, the image of φ would have 8 elements. But the image is a subgroup of of the codomain. This can't happen because $\boxed{|\varphi(D_4)| = 8 \text{ does not divide } |\mathbb{Z}_{12}| = 12}$.

Alternatively, if φ was one-to-one, we would have $D_4 \cong D_4/\{1\} = D_4/\ker(\varphi) \cong \varphi(D_4)$. Thus $\varphi(D_4)$ would be non-Abelian (because it's isomorphic to D_4 which isn't Abelian). But $\varphi(D_4)$ is a subgroup of the codomain $\mathbb{Z}_{12}$. This cannot be because an Abelian group cannot have a non-Abelian subgroup!

(c) The center of $D_6 = \langle x, y \mid x^6 = 1, y^2 = 1, (xy)^2 = 1\rangle = \{1, x, \ldots, x^5, y, xy, \ldots, x^5y\}$ is $Z(D_6) = \{1, x^3\}$. Suppose a classmate thinks they found an element of order 6 in $D_6/Z(D_6)$. How do we know they are wrong?

The G/Z-Theorem states that if $G/Z(G)$ is cyclic, then G must be Abelian. Notice that $\left|D_6/Z(D_6)\right| = \dfrac{|D_6|}{|Z(D_6)|} = \dfrac{12}{2} = 6$. Thus if $D_6/Z(D_6)$ had an element of order 6, it would be cyclic. If it was cyclic, the G/Z-Theorem would imply that D_6 is Abelian – but it's not! Thus $D_6/Z(D_6)$ cannot have any element of order 6. *Note:* The only two groups of order 6 (up to isomorphism) are $\mathbb{Z}_6$ and D_3. This discussion reveals that $D_6/Z(D_6)$ must be isomorphic to D_3.

Alternatively, we could approach this question by direct calculation. Consider a quotient group G/H. Then the order of the coset gH (working in G/H) must divide the order of its representative g (working in G). The elements in D_6 have orders 1, 2, 3, and 6. Thus only elements of order 6 could possibly represent a coset of order 6. The elements of order 6 in D_6 are x and x^5. Notice that $(xZ(D_6))^3 = x^3Z(D_6) = Z(D_6)$ since $x^3 \in Z(D_6)$, so its order is at most 3. Also, $x^5Z(D_6) = x^{-1}Z(D_6) = (xZ(D_6))^{-1}$ so its order is the same (at most 3). Thus we do not have any elements of order 6.

6 (10 points) Let H and K be normal subgroups of G. Prove that $H \cap K$ is a normal subgroup of G.

Note: Show **both** $H \cap K$ is a subgroup **and** that it's normal in G.

We run through the (normal) subgroup test. First, $e \in H$ and $e \in K$ since subgroups must contain the identity element. Therefore, $e \in H \cap K$ (thus the intersect is a non-empty subset). Next, let $a, b \in H \cap K$ and $g \in G$. Then $a \in H$ and $b \in K$. Thus since H is a subgroup, we have both ab and a^{-1} belong to H. Moreover, H is a normal subgroup, so $gag^{-1} \in H$ (i.e., it's closed under conjugation). The same is true for K. Therefore, since ab, a^{-1}, and gag^{-1} belong to both H and K, we have $ab, a^{-1}, gag^{-1} \in H \cap K$ showing the intersection is closed under the operation, inversion, and conjugation. Thus $H \cap K$ is normal in G.

7. (20 points) Finite Abelian Groups

(a) List all of the non-isomorphic Abelian groups of order $200 = 2^3 5^2$. Circle any that are cyclic.

Since there are 3 partitions of 3 (i.e., $3 = 2 + 1 = 1 + 1 + 1$) and 2 partitions of 2 (i.e., $2 = 1 + 1$), we know there will be $6 = 2 \cdot 3$ isomorphism classes of Abelian groups of this order. I will show both the elementary divisor and invariant factor forms of each one. Of course, the only one with elements of 200 (i.e., the first one listed) is cyclic.

- $\boxed{\mathbb{Z}_8 \times \mathbb{Z}_{25} \cong \mathbb{Z}_{200}}$
- $\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_{25} \cong \mathbb{Z}_2 \times \mathbb{Z}_{100}$
- $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{25} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{50}$
- $\mathbb{Z}_8 \times \mathbb{Z}_5 \times \mathbb{Z}_5 \cong \mathbb{Z}_5 \times \mathbb{Z}_{40}$
- $\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_5 \times \mathbb{Z}_5 \cong \mathbb{Z}_{10} \times \mathbb{Z}_{20}$
- $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_5 \times \mathbb{Z}_5 \cong \mathbb{Z}_2 \times \mathbb{Z}_{10} \times \mathbb{Z}_{10}$

(b) Suppose that G is an Abelian group of order 200 and we know it has an element of order 8 but G is not cyclic. Using part (a), which group(s) could G be isomorphic to?

To get an element of order 8, we require one of the elementary divisors to be 8. In other words, only the first and fourth groups fit the bill. Since our group isn't cyclic, it can't be the first one. Therefore, $\boxed{G \cong \mathbb{Z}_8 \times \mathbb{Z}_5 \times \mathbb{Z}_5}$ ($\cong \mathbb{Z}_5 \times \mathbb{Z}_{40}$).

(c) How many non-isomorphic Abelian groups of order 793,800 are there?

Note: $793{,}800 = 2^3 \cdot 3^4 \cdot 5^2 \cdot 7^2$ and there are 5 non-isomorphic Abelian groups of order $81 = 3^4$. ☺

This amounts to the product of the number of partitions of each of the exponents: There are $3 \cdot 5 \cdot 2 \cdot 2 = \boxed{60}$ such group structures.

(d) What is the largest order among elements of $\mathbb{Z}_{10} \times \mathbb{Z}_4 \times \mathbb{Z}_6$? Explain you answer.

The orders in $\mathbb{Z}_{10}$ divide 10, in $\mathbb{Z}_4$ divide 4, and in $\mathbb{Z}_6$ divide 6. Orders in this product group of least common multiples of such divisors. Thus the largest possible order is $\text{lcm}(10, 4, 6) = \boxed{60}$. *Note:* This is achieved by $(1, 1, 1)$.

Alternatively, we could rearrange in terms of invariant factors. First, I'll break apart in terms of prime powers (i.e., elementary divisors). Then, I'll assemble the invariant factors (by taking the largest of each of the distinct prime powers successively): $\mathbb{Z}_{10} \times \mathbb{Z}_4 \times \mathbb{Z}_6 \cong \mathbb{Z}_2 \times \mathbb{Z}_5 \times \mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \cong \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{60}$. Evidently the largest available order (immediately visible from the invariant factors) is 60.