Name ______________________________ Math 351, 03, Exam #1 Oct., 2006

Be sure to show all your work. Unsupported answers will receive no credit.

Use the backs of the exam pages for scratchwork or for continuation of your answers, if necessary.

Problem No.	**Pts Possible**	**Points**
1	10	
2	25	
3	10	
4	10	
5	10	
6	12	
7	12	
8	10	
	+1	
Total	100	

1. (10 points): This is a "gimmie"

(a) Write out the definition of a ring – give me all of the gory details please!

(b) Give the definition of the greatest common divisor of two polynomials $f(x)$ and $g(x)$ in $\mathbb{F}[x]$ where $\mathbb{F}$ is a field (and either $f(x) \neq 0$ or $g(x) \neq 0$).

2. (25 points): Circle "True" if the statement is always true, "Possible" if the statement is true for some examples but false for others, or "False" if the statement is never true. **Then justify your choice!**

(a) Let a, b, u, and v be integers such that $au + bv = 2$.
TRUE / POSSIBLE / FALSE: The greatest common divisor of a and b is 2.

(b) Let $y \in \mathbb{Z}$.
TRUE / POSSIBLE / FALSE: The equation $2x \equiv y \bmod 999$ has a solution $x \in \mathbb{Z}$.

(c) Let n be a positive integer.
TRUE / POSSIBLE / FALSE: Both cancellation laws hold in $\mathbb{Z}_n$.

(d) Let $\mathbb{F}$ be a field.
TRUE / POSSIBLE / FALSE: $\mathbb{F} \cong M(\mathbb{F})$.

(e) Let $\mathbb{F}$ be a field, and let $f(x) \in \mathbb{F}[x]$ be a polynomial of degree 5 such that $f(x)$ has no irreducible factors of degree 3 or 4.
TRUE / POSSIBLE / FALSE: $f(x)$ is irreducible.

3. (10 points): Basics

(a) Let R be a ring with 1. Prove that $1 = 0$ if and only if $R = \{0\}$ (the trivial ring).

(b) Let R be a ring with 1 and $r \in R$. Show that r cannot be both a zero divisor and a unit.

4. (10 points): Choices! Prove ONE of the following:

I. Let R be a ring where $a^2 = a$ for each $a \in R$. Show that R is commutative. *Hint:* For all $a, b \in R$, $(a+b)^2 = (a+b)$ and $(a+a)^2 = (a+a)$.

II. Let R be a ring with 1. Prove that the characteristic of R and and the characteristic of $M(R)$ are the same.

5. (10 points): Let $\phi : R \to S$ be a homomorphism from a ring R to a ring S. Also, let T be a subring of R. Prove that $\phi(T) = \{\phi(x) \in S \,|\, x \in T\}$ is a subring of S.

6. (12 points): Workin' in $\mathbb{Z}_9$.

(a) Write the multiplication table for $\mathbb{Z}_9$.

(b) List all of the members of $U(\mathbb{Z}_9)$.

(c) List all of the zero divisors in $\mathbb{Z}_9$.

(d) Is $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ a unit of $M(\mathbb{Z}_9)$? If so, find its inverse. If not, explain why.

7. (12 points): Isomorphisms.

(a) Is $\mathbb{Q}$ isomorphic to $\mathbb{Z}$? Why or why not?

(b) Is $\mathbb{R}\times\mathbb{R}$ isomorphic to the subring of 2x2 real diagonal matrices $D = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mid a, b \in \mathbb{R} \right\}$? Why or why not?

(c) Is $M(\mathbb{Z}_2)$ isomorphic to $\mathbb{Z}_{16}$? Why or why not?

8. (10 points): Factoring in $\mathbb{Z}_3[x]$.

(a) List all of the monic polynomials of degree 2 in $\mathbb{Z}_3[x]$. Circle all of the irreducible polynomials in your list.

(b) Show that $f(x) = x^5 + x^4 + 2x^3 + x^2 + x + 1$ is irreducible in $\mathbb{Z}_3[x]$.