Name ______________________________ Math 351, 03, Exam #2 Nov., 2006

Be sure to show all your work. Unsupported answers will receive no credit.

Use the backs of the exam pages for scratchwork or for continuation of your answers, if necessary.

Problem No.	**Pts Possible**	**Points**
1	20	
2	12	
3	10	
4	12	
5	12	
6	12	
7	12	
8	10	
Total	100	

1. (20 points): Circle "True" if the statement is always true, "Possible" if the statement is true for some examples but false for others, or "False" if the statement is never true. **Then justify your choice!**

(a) Let R be a finite integral domain.
TRUE / POSSIBLE / FALSE: R has no irreducible elements.

(b) Let R be a unique factorization domain.
TRUE / POSSIBLE / FALSE: R is a Euclidean domain.

(c) Let R be an integral domain, and let $p \in R$ such that (p) is a prime ideal of R.
TRUE / POSSIBLE / FALSE: p is a prime in R.

(d) Consider the quotient ring $R = \mathbb{Z}[x]/(x^2 + 1)$.
TRUE / POSSIBLE / FALSE: R has the ascending chain condition.

(e) Let R be a finite integral domain.
TRUE / POSSIBLE / FALSE: The field of quotients of R is bigger than R.

2. (12 points): Euclidean domains.

(a) Give the definition of a "Euclidean domain".

(b) Fix a non-negative integer integer c. Let R be a Euclidean domain such that $\delta(x) = c$ for all $x \in R$, $x \neq 0$. Prove the R is a field.

(c) Let R be a Euclidean domain. Show that $\delta(x) \geq \delta(1)$ for all nonzero $x \in R$.

3. (10 points): An ideal question.

(a) Is there a homomorphism whose domain is $\mathbb{Z}[x]$ and kernel is $(x^2 - 4)$? If so, find one. If not, explain why.

(b) Is there a homomorphism whose domain is $M(\mathbb{R}) = \mathbb{R}^{2\times 2}$ and kernel is $D = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \middle| \, a, b \in \mathbb{R} \right\}$? If so, find one. If not, explain why.

4. (12 points): Fielding questions.

(a) Construct a field with 9 elements.

(b) Consider $f(y) = y^3 + y \in \mathbb{Z}_3[y]$. Find all of the roots of $f(y)$ in your field of 9 elements.

5. (12 points): Let $I = (x^2 - 4)$ be a principal ideal in $\mathbb{Q}[x]$.

(a) Is $x^2 + x + I$ a unit in $\mathbb{Q}[x]/I$? If not, explain why. If so, find its inverse.

(b) Does $\mathbb{Q}[x]/I$ have any zero divisors? If not, explain why. If so, find one.

(c) I is not a maximal ideal of $\mathbb{Q}[x]$. Use part (b) to explain why. Also, find an ideal J of $\mathbb{Q}[x]$ so that $I \subsetneq J \subsetneq \mathbb{Q}[x]$.

6. (12 points): The Third Isomorphism Theorem – Let I and J be ideals of R where $J \subseteq I$.

(a) Show that I/J is an ideal of R/J directly (i.e. **Do not find a homomorphism whose kernel is** I/J).

(b) Oh horrors! A quotient of quotients!
Show that $(R/J)/(I/J)$ is isomorphic to R/I. *Hint:* Use the first isomorphism theorem.

7. (12 points): Isomorphic or not isomorphic?

(a) Is $\mathbb{Z}[x]$ isomorphic to $\mathbb{Z}[i]$? Why or why not?

(b) Is $\mathbb{Q}[x]/(x+2)$ isomorphic to $\mathbb{Q}$? Why or why not?

(c) Is $\mathbb{Z}$ isomorphic to $\mathbb{Q}[x]/(f(x))$ for some $f(x) \in \mathbb{Q}[x]$? Why or why not?

8. (10 points): Fields of quotients.

(a) Show that $\mathbb{Q}[\sqrt{-5}]$ is a field.

(b) Prove that $\mathbb{Q}[\sqrt{-5}]$ is the field of quotients of $\mathbb{Z}[\sqrt{-5}]$.